Lecture 13: The Cross Product and the Camera

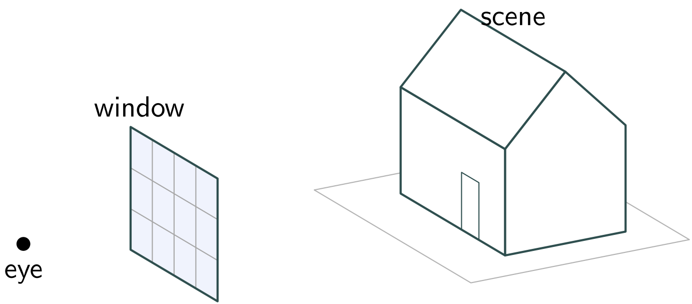
A Frame, a Window, and One Ray per Pixel

CS 453 — Computer Graphics

Fakhir Shaheen

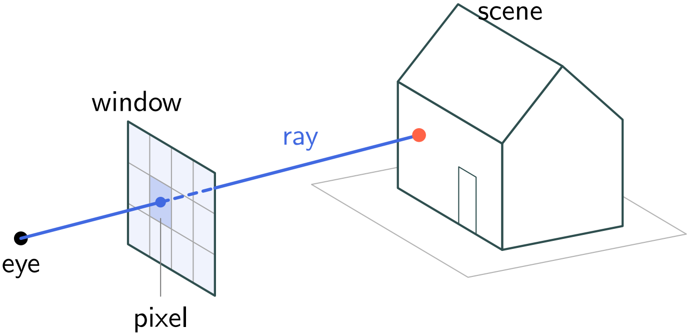
Forman Christian University

Today: one ray through every pixel



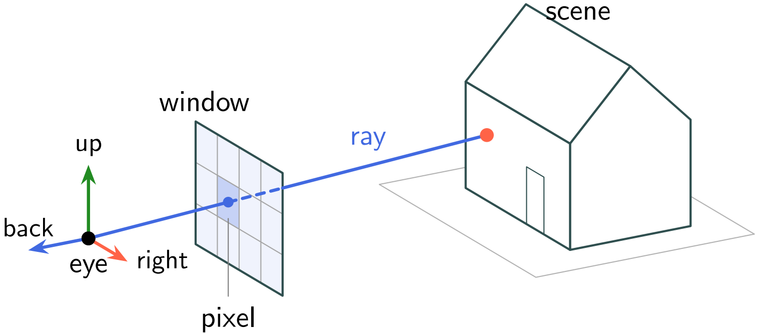
- An eye, a window, a scene

Today: one ray through every pixel



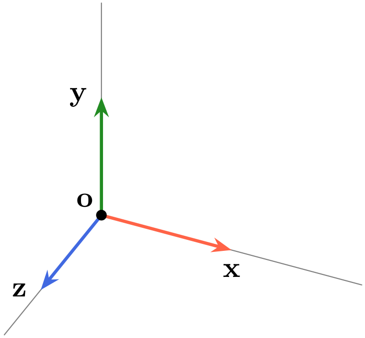
- An eye, a window, a scene
- One ray through every pixel

Today: one ray through every pixel



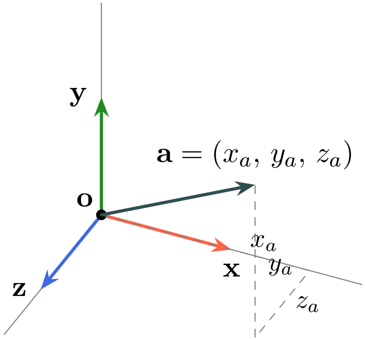
- An eye, a window, a scene
- One ray through every pixel
- First: the camera's own directions

Our conventions in one picture



x right, y up, z toward you

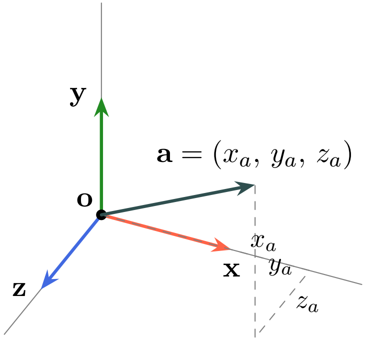
Our conventions in one picture

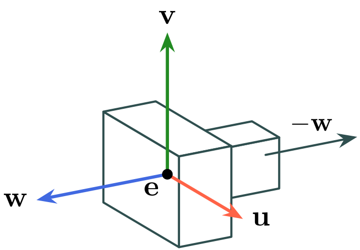


x right, y up, z toward you

Bold: arrow or point

Our conventions in one picture



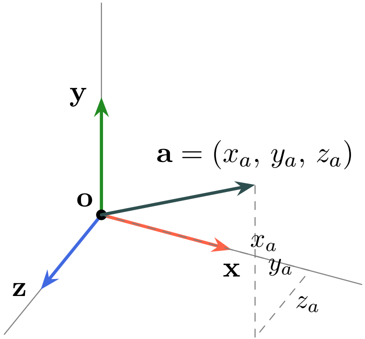


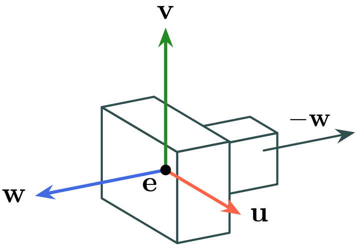
x right, y up, z toward you

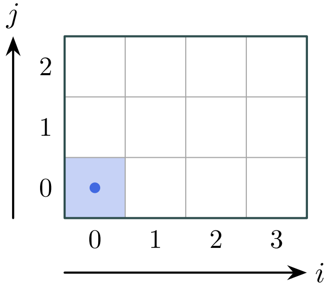
Camera looks along $-\mathbf{w}$

Bold: arrow or point

Our conventions in one picture







x right, y up, z toward you

Bold: arrow or point

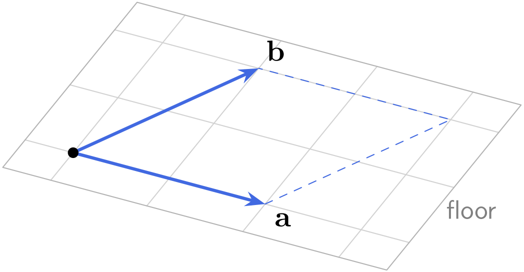
Camera looks along $-\mathbf{w}$

Pixel $(0, 0)$: bottom left

The cross product

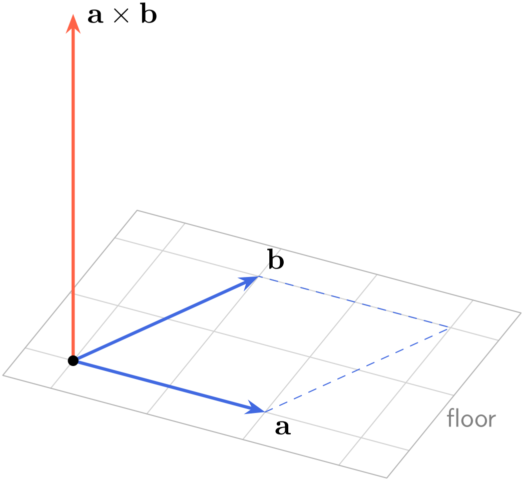
Two arrows in, one perpendicular arrow out

The cross product is perpendicular to both



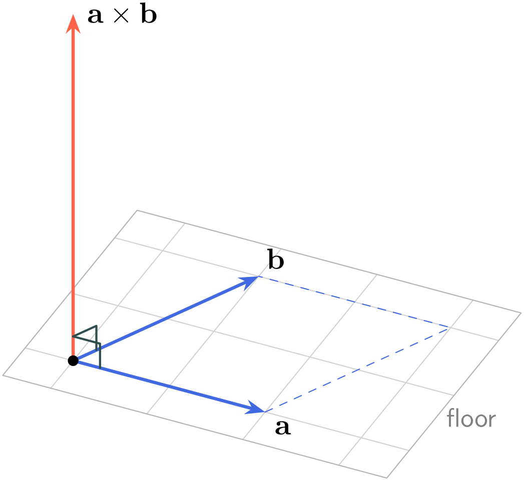
- Two 3D arrows go in

The cross product is perpendicular to both



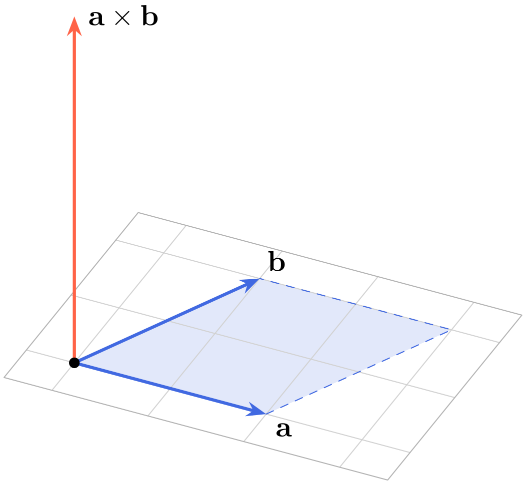
- Two 3D arrows go in
- One 3D arrow comes out

The cross product is perpendicular to both



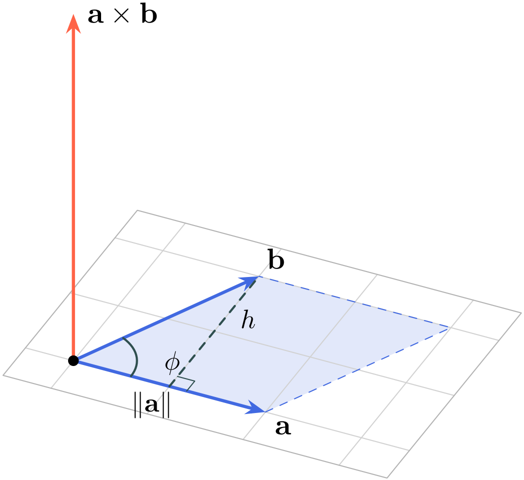
- Two 3D arrows go in
- One 3D arrow comes out
- At right angles to both

Its length is the parallelogram's area



- The arrow's length is this area

Its length is the parallelogram's area

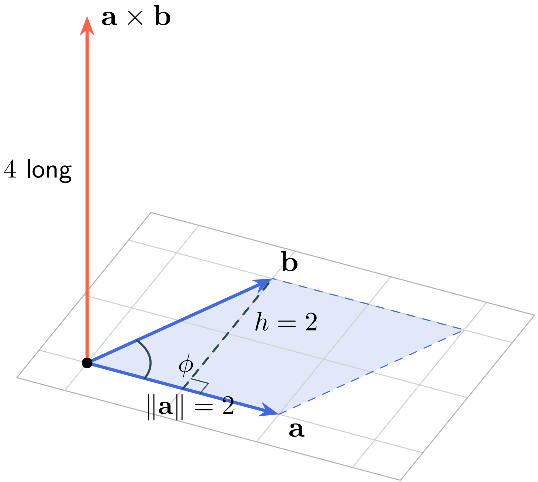


- The arrow's length is this area
- Area: base times height

  $h = \|\mathbf{b}\| \sin\phi$

  $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\phi$

Its length is the parallelogram's area

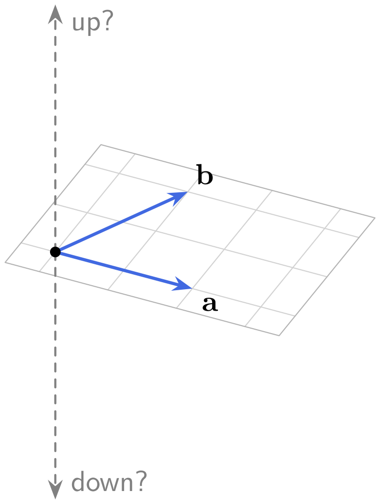


- The arrow's length is this area
- Area: base times height

  $h = \|\mathbf{b}\| \sin\phi$

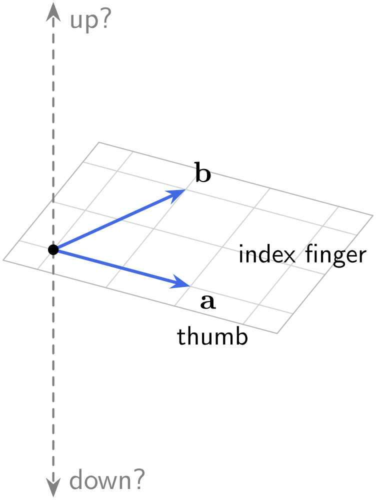
  $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\phi$
- Here: 2 by 2, so 4

Two directions fit; the right hand picks



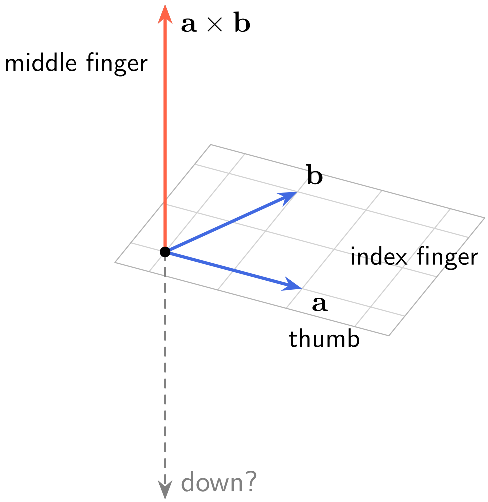
- Up or down: both fit

Two directions fit; the right hand picks



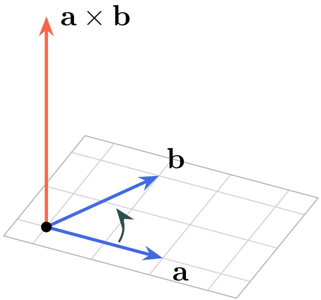
- Up or down: both fit
- Thumb on $\mathbf{a}$, index finger on $\mathbf{b}$

Two directions fit; the right hand picks



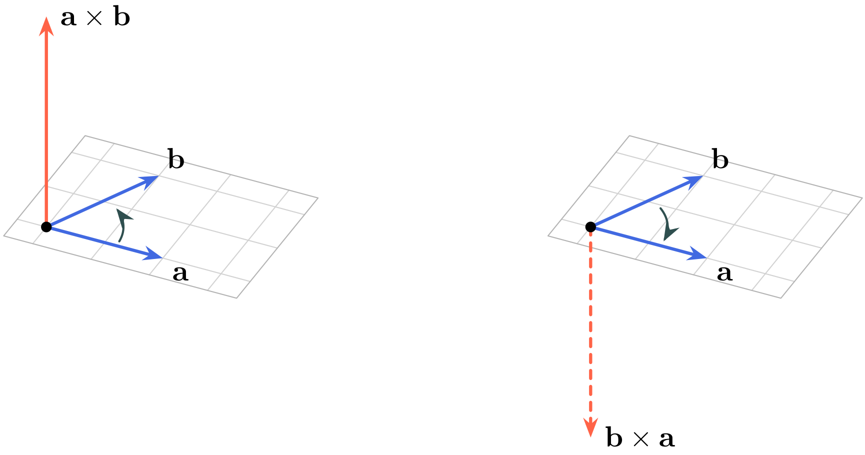
- Up or down: both fit
- Thumb on $\mathbf{a}$, index finger on $\mathbf{b}$
- Middle finger gives $\mathbf{a} \times \mathbf{b}$

Swap the order, flip the arrow



Same two arrows

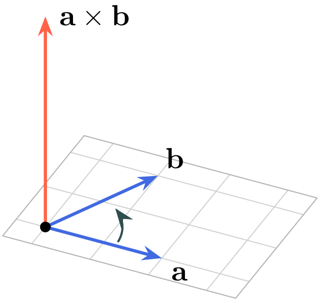
Swap the order, flip the arrow



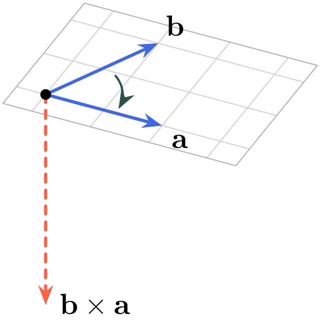
Same two arrows

Same length, opposite direction

Swap the order, flip the arrow



$$\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$$

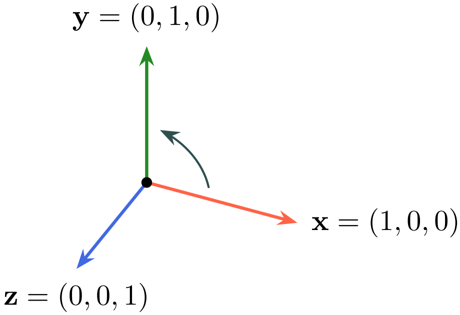


Same two arrows

Same length, opposite direction

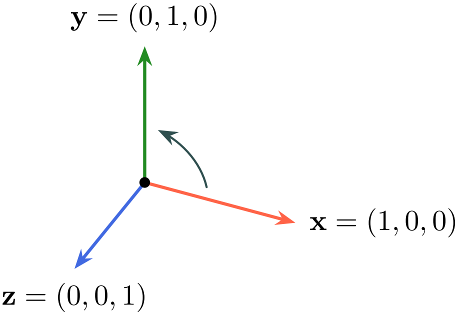
Order matters

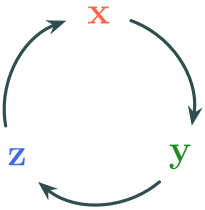
The three axes cross in a cycle



- Our choice: $\mathbf{z} = \mathbf{x} \times \mathbf{y}$

The three axes cross in a cycle





the wheel

- With the wheel: plus

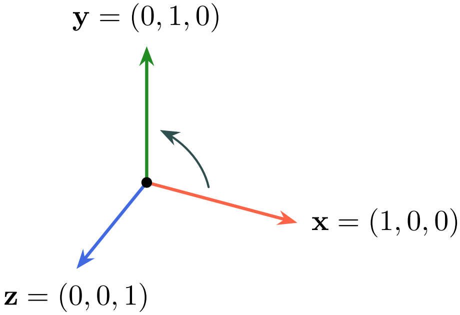
$$\mathbf{x} \times \mathbf{y} = +\mathbf{z}$$

$$\mathbf{y} \times \mathbf{z} = +\mathbf{x}$$

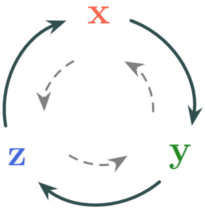
$$\mathbf{z} \times \mathbf{x} = +\mathbf{y}$$

- Our choice: $\mathbf{z} = \mathbf{x} \times \mathbf{y}$

The three axes cross in a cycle



- Our choice: $\mathbf{z} = \mathbf{x} \times \mathbf{y}$



the wheel

- With the wheel: plus

$\mathbf{x} \times \mathbf{y} = +\mathbf{z}$

$\mathbf{y} \times \mathbf{z} = +\mathbf{x}$

$\mathbf{z} \times \mathbf{x} = +\mathbf{y}$

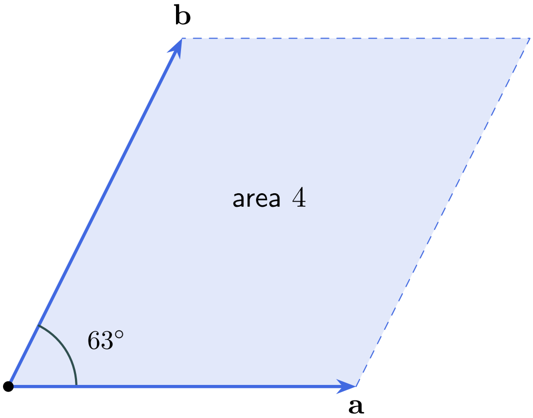
- Against the wheel: minus

$\mathbf{y} \times \mathbf{x} = -\mathbf{z}$

$\mathbf{z} \times \mathbf{y} = -\mathbf{x}$

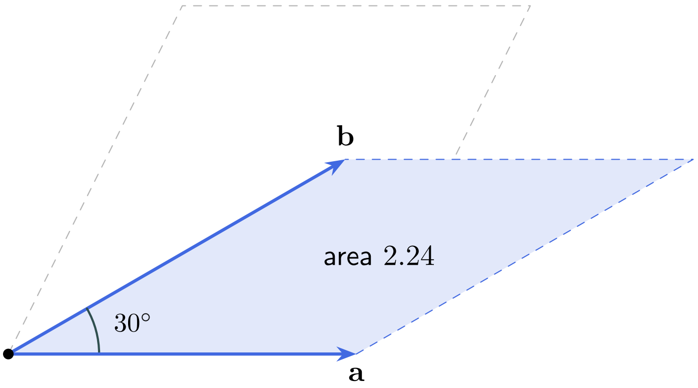
$\mathbf{x} \times \mathbf{z} = -\mathbf{y}$

A vector crossed with itself is zero



the floor, seen from above

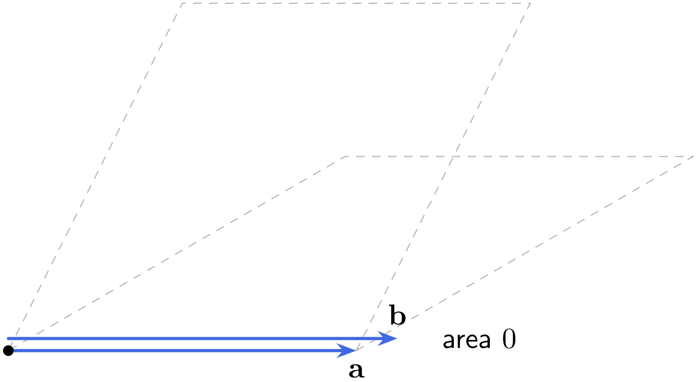
A vector crossed with itself is zero



the floor, seen from above

- Close the angle: area shrinks

A vector crossed with itself is zero



the floor, seen from above

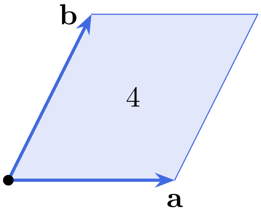
- Close the angle: area shrinks
- Same line: area zero

$$\mathbf{x} \times \mathbf{x} = \mathbf{0}$$

the zero vector

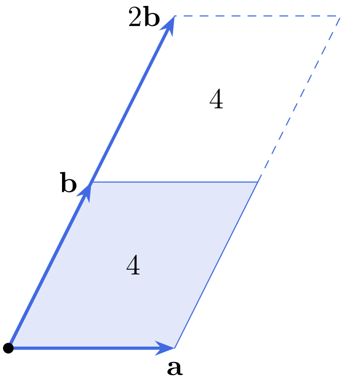
Two rules let us multiply out

the floor, seen from above



Two rules let us multiply out

the floor, seen from above

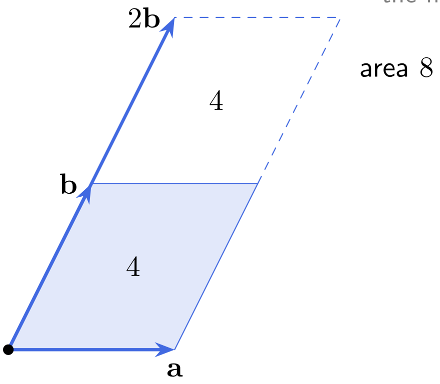


area 8

$$\mathbf{a} \times (k\mathbf{b}) = k\,(\mathbf{a} \times \mathbf{b})$$

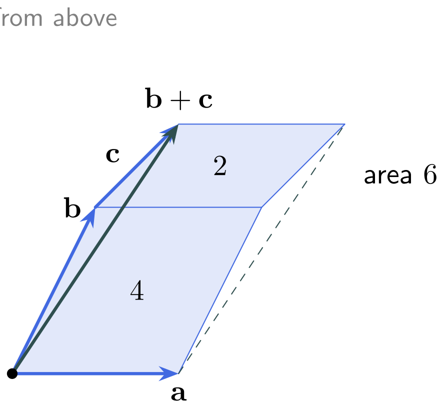
Stretch $\mathbf{b}$: the area stretches

Two rules let us multiply out



$$\mathbf{a} \times (k\mathbf{b}) = k\,(\mathbf{a} \times \mathbf{b})$$

Stretch $\mathbf{b}$: the area stretches



$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$$

Add $\mathbf{c}$: the areas add

Multiplying out gives nine small products

$$\mathbf{a} \times \mathbf{b} = (x_a\mathbf{x} + y_a\mathbf{y} + z_a\mathbf{z}) \times (x_b\mathbf{x} + y_b\mathbf{y} + z_b\mathbf{z})$$

Multiplying out gives nine small products

$$\mathbf{a} \times \mathbf{b} = (x_a\mathbf{x} + y_a\mathbf{y} + z_a\mathbf{z}) \times (x_b\mathbf{x} + y_b\mathbf{y} + z_b\mathbf{z})$$

$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$
$x_a\mathbf{x}$	$x_a x_b\ \mathbf{x} \times \mathbf{x}$	$x_a y_b\ \mathbf{x} \times \mathbf{y}$	$x_a z_b\ \mathbf{x} \times \mathbf{z}$
$y_a\mathbf{y}$			
$z_a\mathbf{z}$			

- Every part meets every part (sum rule)
- Numbers move to the front (stretch rule)

Multiplying out gives nine small products

$$\mathbf{a} \times \mathbf{b} = (x_a\mathbf{x} + y_a\mathbf{y} + z_a\mathbf{z}) \times (x_b\mathbf{x} + y_b\mathbf{y} + z_b\mathbf{z})$$

$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$
$x_a\mathbf{x}$	$x_a x_b\ \mathbf{x} \times \mathbf{x}$	$x_a y_b\ \mathbf{x} \times \mathbf{y}$	$x_a z_b\ \mathbf{x} \times \mathbf{z}$
$y_a\mathbf{y}$	$y_a x_b\ \mathbf{y} \times \mathbf{x}$	$y_a y_b\ \mathbf{y} \times \mathbf{y}$	$y_a z_b\ \mathbf{y} \times \mathbf{z}$
$z_a\mathbf{z}$			

- Every part meets every part (sum rule)
- Numbers move to the front (stretch rule)

Multiplying out gives nine small products

$$\mathbf{a} \times \mathbf{b} = (x_a\mathbf{x} + y_a\mathbf{y} + z_a\mathbf{z}) \times (x_b\mathbf{x} + y_b\mathbf{y} + z_b\mathbf{z})$$

$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$
$x_a\mathbf{x}$	$x_a x_b\ \mathbf{x} \times \mathbf{x}$	$x_a y_b\ \mathbf{x} \times \mathbf{y}$	$x_a z_b\ \mathbf{x} \times \mathbf{z}$
$y_a\mathbf{y}$	$y_a x_b\ \mathbf{y} \times \mathbf{x}$	$y_a y_b\ \mathbf{y} \times \mathbf{y}$	$y_a z_b\ \mathbf{y} \times \mathbf{z}$
$z_a\mathbf{z}$	$z_a x_b\ \mathbf{z} \times \mathbf{x}$	$z_a y_b\ \mathbf{z} \times \mathbf{y}$	$z_a z_b\ \mathbf{z} \times \mathbf{z}$

- Every part meets every part (sum rule)
- Numbers move to the front (stretch rule)

Three products vanish, six pair up

$\mathbf{a} \times \mathbf{b}$ is the sum of the nine cells

$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$
$x_a\mathbf{x}$	$x_a x_b\ \mathbf{x} \times \mathbf{x}$	$x_a y_b\ \mathbf{x} \times \mathbf{y}$	$x_a z_b\ \mathbf{x} \times \mathbf{z}$
$y_a\mathbf{y}$	$y_a x_b\ \mathbf{y} \times \mathbf{x}$	$y_a y_b\ \mathbf{y} \times \mathbf{y}$	$y_a z_b\ \mathbf{y} \times \mathbf{z}$
$z_a\mathbf{z}$	$z_a x_b\ \mathbf{z} \times \mathbf{x}$	$z_a y_b\ \mathbf{z} \times \mathbf{y}$	$z_a z_b\ \mathbf{z} \times \mathbf{z}$

Three products vanish, six pair up

$\mathbf{a} \times \mathbf{b}$ is the sum of the nine cells

- Same axis twice: zero

$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$
$x_a\mathbf{x}$	0	$x_a y_b\ \mathbf{x} \times \mathbf{y}$	$x_a z_b\ \mathbf{x} \times \mathbf{z}$
$y_a\mathbf{y}$	$y_a x_b\ \mathbf{y} \times \mathbf{x}$	0	$y_a z_b\ \mathbf{y} \times \mathbf{z}$
$z_a\mathbf{z}$	$z_a x_b\ \mathbf{z} \times \mathbf{x}$	$z_a y_b\ \mathbf{z} \times \mathbf{y}$	0

Three products vanish, six pair up

$\mathbf{a} \times \mathbf{b}$ is the sum of the nine cells

- Same axis twice: zero

$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$	
$x_a\mathbf{x}$	0	$x_a y_b\ \mathbf{x} \times \mathbf{y}$	$x_a z_b\ \mathbf{x} \times \mathbf{z}$	
$y_a\mathbf{y}$	$y_a x_b\ \mathbf{y} \times \mathbf{x}$	0	$+\,y_a z_b\ \mathbf{x}$	$\mathbf{y} \times \mathbf{z} = \mathbf{x}$
$z_a\mathbf{z}$	$z_a x_b\ \mathbf{z} \times \mathbf{x}$	$-\,z_a y_b\ \mathbf{x}$	0	

- Each axis collects two cells

Three products vanish, six pair up

$\mathbf{a} \times \mathbf{b}$ is the sum of the nine cells

- Same axis twice: zero

$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$	
$x_a\mathbf{x}$	0	$x_a y_b\ \mathbf{x} \times \mathbf{y}$	$-\,x_a z_b\ \mathbf{y}$	
$y_a\mathbf{y}$	$y_a x_b\ \mathbf{y} \times \mathbf{x}$	0	$+\,y_a z_b\ \mathbf{x}$	$\mathbf{y} \times \mathbf{z} = \mathbf{x}$
$z_a\mathbf{z}$	$+\,z_a x_b\ \mathbf{y}$	$-\,z_a y_b\ \mathbf{x}$	0	$\mathbf{z} \times \mathbf{x} = \mathbf{y}$

- Each axis collects two cells

Three products vanish, six pair up

$\mathbf{a} \times \mathbf{b}$ is the sum of the nine cells

- Same axis twice: zero

$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$	
$x_a\mathbf{x}$	0	$+x_a y_b\ \mathbf{z}$	$-x_a z_b\ \mathbf{y}$	$\mathbf{x} \times \mathbf{y} = \mathbf{z}$
$y_a\mathbf{y}$	$-y_a x_b\ \mathbf{z}$	0	$+y_a z_b\ \mathbf{x}$	$\mathbf{y} \times \mathbf{z} = \mathbf{x}$
$z_a\mathbf{z}$	$+z_a x_b\ \mathbf{y}$	$-z_a y_b\ \mathbf{x}$	0	$\mathbf{z} \times \mathbf{x} = \mathbf{y}$

- Each axis collects two cells

Three products vanish, six pair up

$\mathbf{a} \times \mathbf{b}$ is the sum of the nine cells

- Same axis twice: zero

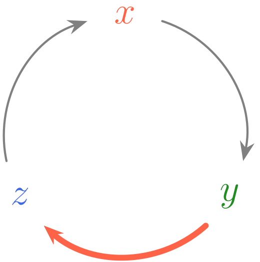
$\times$	$x_b\mathbf{x}$	$y_b\mathbf{y}$	$z_b\mathbf{z}$	
$x_a\mathbf{x}$	0	$+\,x_a y_b\ \mathbf{z}$	$-\,x_a z_b\ \mathbf{y}$	$\mathbf{x} \times \mathbf{y} = \mathbf{z}$
$y_a\mathbf{y}$	$-\,y_a x_b\ \mathbf{z}$	0	$+\,y_a z_b\ \mathbf{x}$	$\mathbf{y} \times \mathbf{z} = \mathbf{x}$
$z_a\mathbf{z}$	$+\,z_a x_b\ \mathbf{y}$	$-\,z_a y_b\ \mathbf{x}$	0	$\mathbf{z} \times \mathbf{x} = \mathbf{y}$

$$\mathbf{a} \times \mathbf{b} = (y_a z_b - z_a y_b)\,\mathbf{x} + (z_a x_b - x_a z_b)\,\mathbf{y} + (x_a y_b - y_a x_b)\,\mathbf{z}$$

- Each axis collects two cells

The cross product, slot by slot

$$\mathbf{a} \times \mathbf{b} = \left(y_a z_b - z_a y_b, \qquad \qquad \qquad \qquad \right)$$

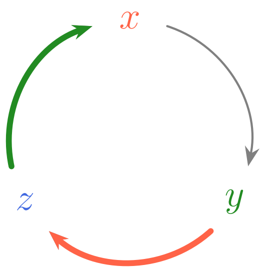


x slot: $y_a z_b - z_a y_b$ after x: y, z

- A slot skips its own letter
- Wheel order first, swapped order second

The cross product, slot by slot

$$\mathbf{a} \times \mathbf{b} = \big(y_a z_b - z_a y_b, \quad z_a x_b - x_a z_b, \qquad\qquad \big)$$



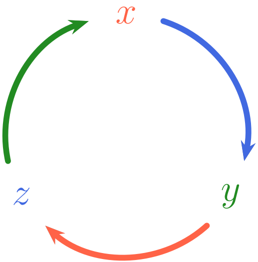
x slot: $y_a z_b - z_a y_b$ after x: y, z

y slot: $z_a x_b - x_a z_b$ after y: z, x

- A slot skips its own letter
- Wheel order first, swapped order second

The cross product, slot by slot

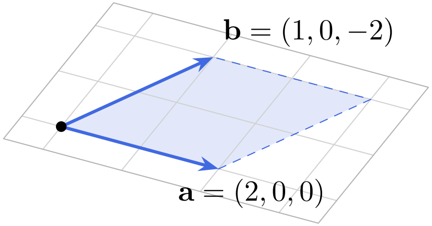
$$\mathbf{a} \times \mathbf{b} = \left(y_a z_b - z_a y_b,\;\; z_a x_b - x_a z_b,\;\; x_a y_b - y_a x_b \right)$$



x slot:	$y_a z_b - z_a y_b$	after x: y, z
y slot:	$z_a x_b - x_a z_b$	after y: z, x
z slot:	$x_a y_b - y_a x_b$	after z: x, y

- A slot skips its own letter
- Wheel order first, swapped order second

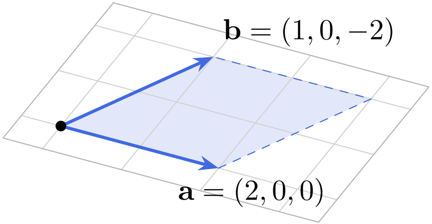
Worked: three slots, one number each



Worked: three slots, one number each

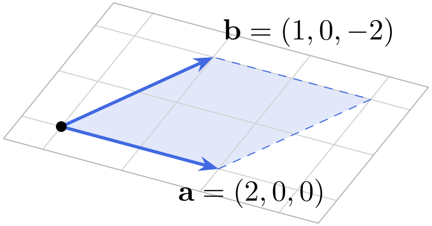
x slot: $y_a z_b - z_a y_b$

$$= (0)(-2) - (0)(0) = 0 - 0 = 0$$



- Per slot: two products, one difference

Worked: three slots, one number each



x slot: $y_a z_b - z_a y_b$

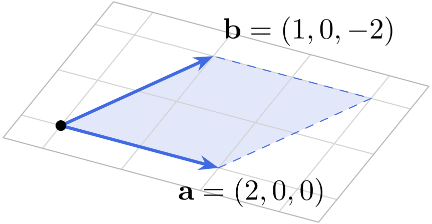
$$= (0)(-2) - (0)(0) = 0 - 0 = 0$$

y slot: $z_a x_b - x_a z_b$

$$= (0)(1) - (2)(-2) = 0 - (-4) = 4$$

- Per slot: two products, one difference

Worked: three slots, one number each



x slot: $y_a z_b - z_a y_b$

$$= (0)(-2) - (0)(0) = 0 - 0 = 0$$

y slot: $z_a x_b - x_a z_b$

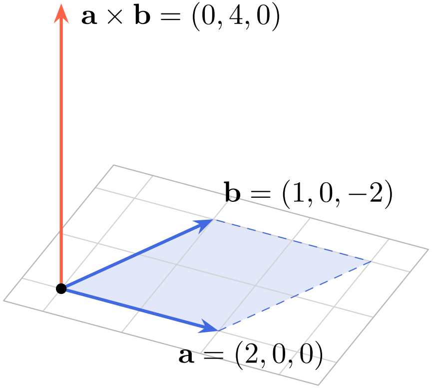
$$= (0)(1) - (2)(-2) = 0 - (-4) = 4$$

z slot: $x_a y_b - y_a x_b$

$$= (2)(0) - (0)(1) = 0 - 0 = 0$$

- Per slot: two products, one difference

Worked: three slots, one number each



x slot: $y_a z_b - z_a y_b$

$$= (0)(-2) - (0)(0) = 0 - 0 = 0$$

y slot: $z_a x_b - x_a z_b$

$$= (0)(1) - (2)(-2) = 0 - (-4) = 4$$

z slot: $x_a y_b - y_a x_b$

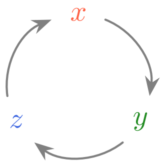
$$= (2)(0) - (0)(1) = 0 - 0 = 0$$

$$\mathbf{a} \times \mathbf{b} = (0,\ 4,\ 0)$$

- Per slot: two products, one difference
- Straight up, 4 long: as drawn

Quick check 1: fill the three slots

	x	y	z
$\mathbf{a}$	1	2	0
$\mathbf{b}$	2	2	-1
$\mathbf{a} \times \mathbf{b}$	?	?	?



- Find $\mathbf{a} \times \mathbf{b}$
- Then $\mathbf{b} \times \mathbf{a}$

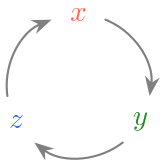
x slot: $y_a z_b - z_a y_b$

y slot: $z_a x_b - x_a z_b$

z slot: $x_a y_b - y_a x_b$

Quick check 1: fill the three slots

	x	y	z
$\mathbf{a}$	1	2	0
$\mathbf{b}$	2	2	-1
$\mathbf{a} \times \mathbf{b}$	-2	1	-2



- Find $\mathbf{a} \times \mathbf{b}$
- Then $\mathbf{b} \times \mathbf{a}$

x slot: $y_a z_b - z_a y_b$

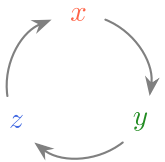
y slot: $z_a x_b - x_a z_b$

z slot: $x_a y_b - y_a x_b$

Quick check 1: fill the three slots

	x	y	z
$\mathbf{a}$	1	2	0
$\mathbf{b}$	2	2	-1
$\mathbf{a} \times \mathbf{b}$	-2	1	-2

$$\mathbf{b} \times \mathbf{a} = (2,\ -1,\ 2)$$



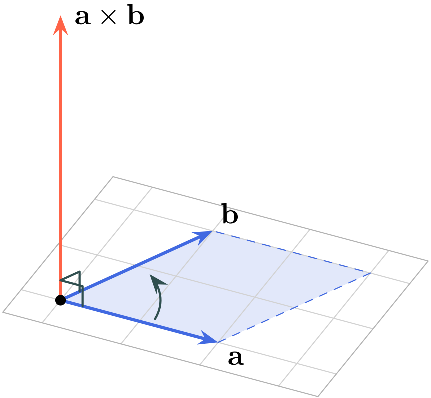
- Find $\mathbf{a} \times \mathbf{b}$
- Then $\mathbf{b} \times \mathbf{a}$

x slot: $y_a z_b - z_a y_b$

y slot: $z_a x_b - x_a z_b$

z slot: $x_a y_b - y_a x_b$

Recap: the cross product

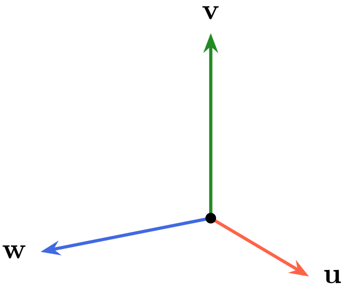


$$\mathbf{a} \times \mathbf{b} = \left(y_a z_b - z_a y_b, \;\; z_a x_b - x_a z_b, \;\; x_a y_b - y_a x_b \right)$$

Building a frame

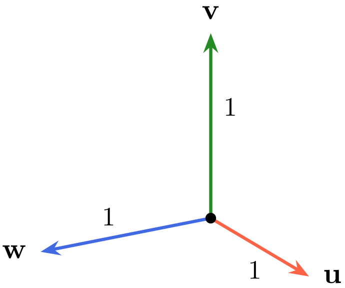
From two arrows to three axes

In 3D a basis has three arrows



- Lecture 9's basis, plus $\mathbf{w}$

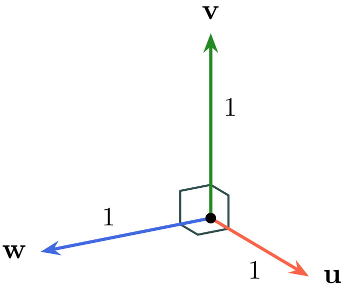
In 3D a basis has three arrows



- Lecture 9's basis, plus $\mathbf{w}$
- Every arrow: length 1

  $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$

In 3D a basis has three arrows

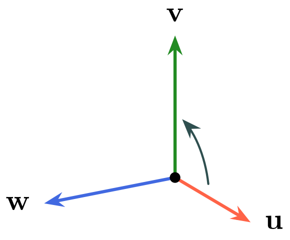


- Lecture 9's basis, plus $\mathbf{w}$
- Every arrow: length 1

  $\|\mathbf{u}\| = \|\mathbf{v}\| = \|\mathbf{w}\| = 1$
- Every pair: a right angle

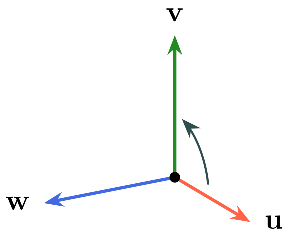
  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{u} = 0$

Right-handed means $\mathbf{w} = \mathbf{u} \times \mathbf{v}$

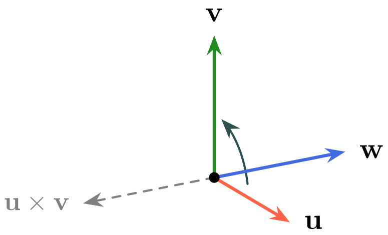


$\mathbf{w} = \mathbf{u} \times \mathbf{v}$: right-handed

Right-handed means $\mathbf{w} = \mathbf{u} \times \mathbf{v}$



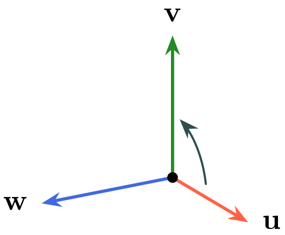
$\mathbf{w} = \mathbf{u} \times \mathbf{v}$: right-handed

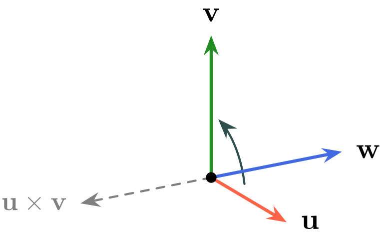


$\mathbf{w} = -(\mathbf{u} \times \mathbf{v})$: left-handed

Same $\mathbf{u}$, same $\mathbf{v}$ $\quad$ $\mathbf{w}$ flipped: left-handed

Right-handed means $\mathbf{w} = \mathbf{u} \times \mathbf{v}$





✓ $\mathbf{w} = \mathbf{u} \times \mathbf{v}$: right-handed

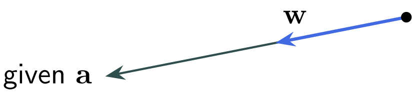
$\mathbf{w} = -(\mathbf{u} \times \mathbf{v})$: left-handed

Same $\mathbf{u}$, same $\mathbf{v}$

$\mathbf{w}$ flipped: left-handed

We build right-handed

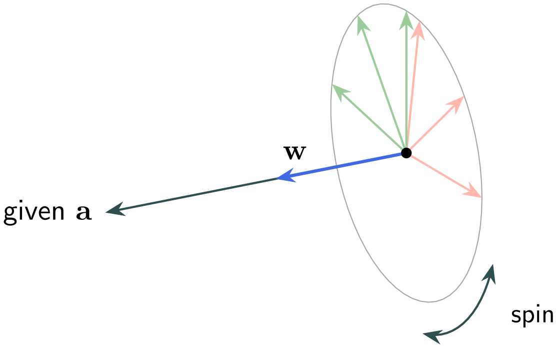
Goal: three axes from one given arrow



- Wanted: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, with $\mathbf{w}$ along $\mathbf{a}$
- $\mathbf{w}$: $\mathbf{a}$ cut to length 1

$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

Goal: three axes from one given arrow

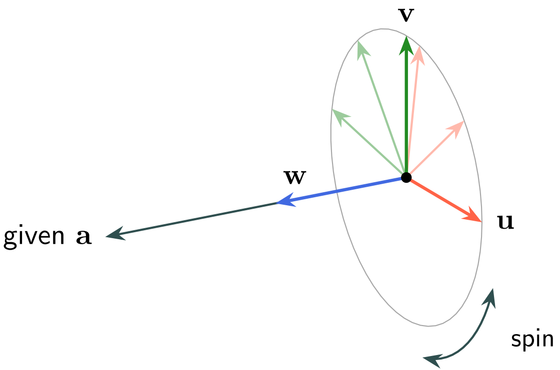


- Wanted: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, with $\mathbf{w}$ along $\mathbf{a}$
- $\mathbf{w}$: $\mathbf{a}$ cut to length 1

  $$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

- $\mathbf{u}$, $\mathbf{v}$: many pairs fit

Goal: three axes from one given arrow

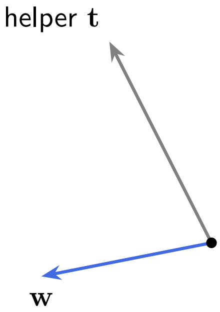


- Wanted: $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, with $\mathbf{w}$ along $\mathbf{a}$
- $\mathbf{w}$: $\mathbf{a}$ cut to length 1

$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

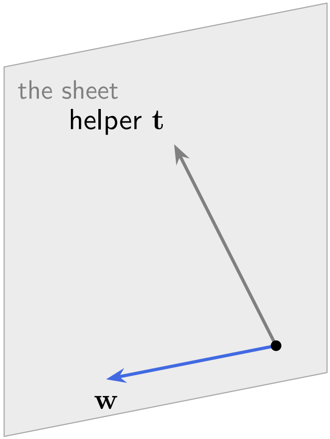
- $\mathbf{u}$, $\mathbf{v}$: many pairs fit
- Still needed: a rule to pick one

A helper arrow completes the basis



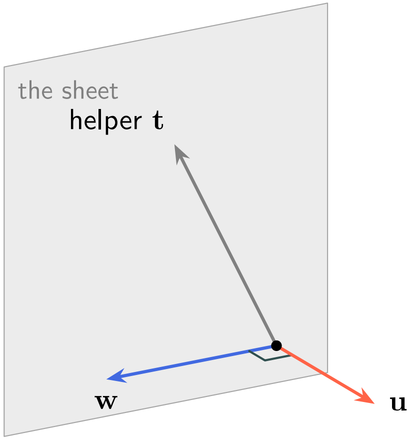
- Helper $\mathbf{t}$: any arrow not along $\mathbf{w}$

A helper arrow completes the basis



- Helper $\mathbf{t}$: any arrow not along $\mathbf{w}$
- $\mathbf{w}$ and $\mathbf{t}$ lie on one flat sheet

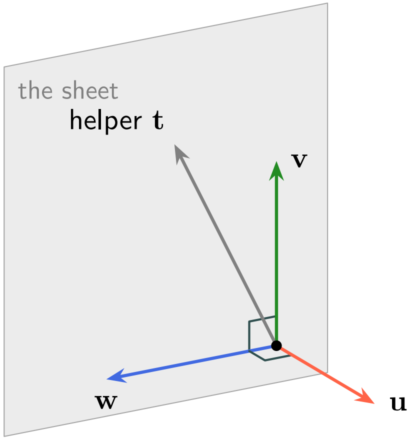
A helper arrow completes the basis



- Helper $\mathbf{t}$: any arrow not along $\mathbf{w}$
- $\mathbf{w}$ and $\mathbf{t}$ lie on one flat sheet
- $\mathbf{u}$ stands perpendicular to the sheet

$$\mathbf{u} = \frac{\mathbf{t} \times \mathbf{w}}{\|\mathbf{t} \times \mathbf{w}\|}$$

A helper arrow completes the basis



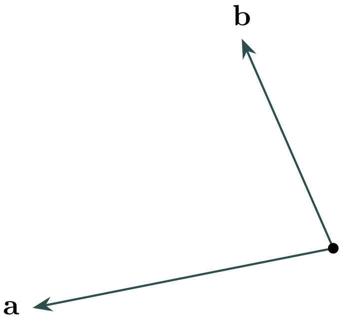
- Helper $\mathbf{t}$: any arrow not along $\mathbf{w}$
- $\mathbf{w}$ and $\mathbf{t}$ lie on one flat sheet
- $\mathbf{u}$ stands perpendicular to the sheet

  $$\mathbf{u} = \frac{\mathbf{t} \times \mathbf{w}}{\|\mathbf{t} \times \mathbf{w}\|}$$

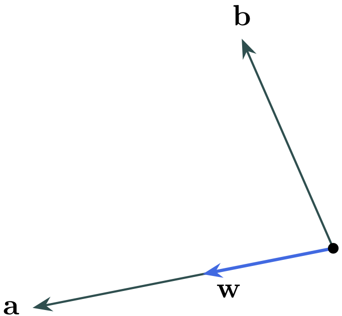
- $\mathbf{v}$ lies on the sheet, perpendicular to $\mathbf{w}$

  $$\mathbf{v} = \mathbf{w} \times \mathbf{u}$$

Two arrows pin the whole frame



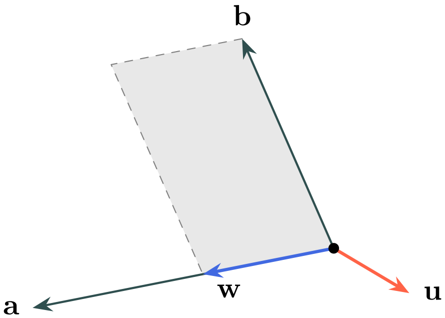
Two arrows pin the whole frame



- $\mathbf{a}$ aims $\mathbf{w}$

$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

Two arrows pin the whole frame



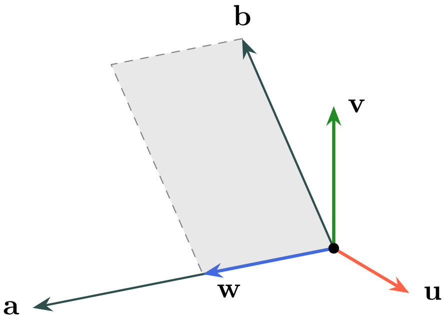
- $\mathbf{a}$ aims $\mathbf{w}$

$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

- $\mathbf{b}$ and $\mathbf{w}$ give $\mathbf{u}$

$$\mathbf{u} = \frac{\mathbf{b} \times \mathbf{w}}{\|\mathbf{b} \times \mathbf{w}\|}$$

Two arrows pin the whole frame



- $\mathbf{a}$ aims $\mathbf{w}$

$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

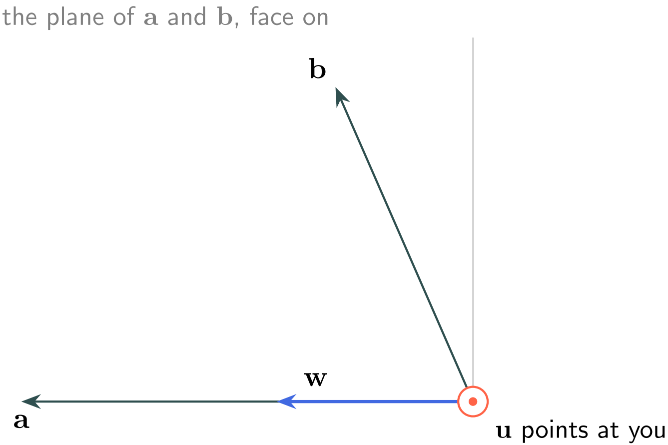
- $\mathbf{b}$ and $\mathbf{w}$ give $\mathbf{u}$

$$\mathbf{u} = \frac{\mathbf{b} \times \mathbf{w}}{\|\mathbf{b} \times \mathbf{w}\|}$$

- $\mathbf{w}$ and $\mathbf{u}$ give $\mathbf{v}$

$$\mathbf{v} = \mathbf{w} \times \mathbf{u}$$

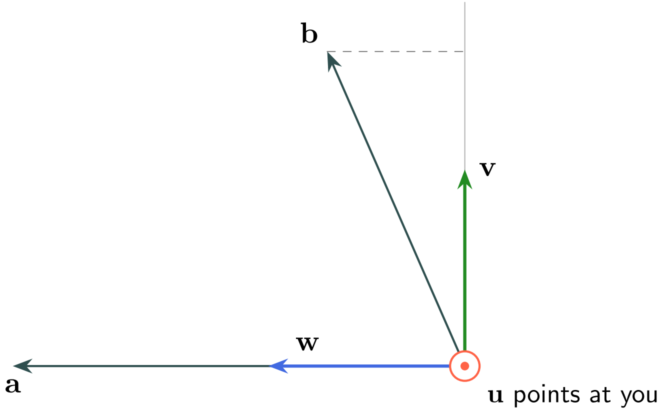
$\mathbf{v}$ is $\mathbf{b}$, straightened up



- $\mathbf{w}$ stays exactly on $\mathbf{a}$

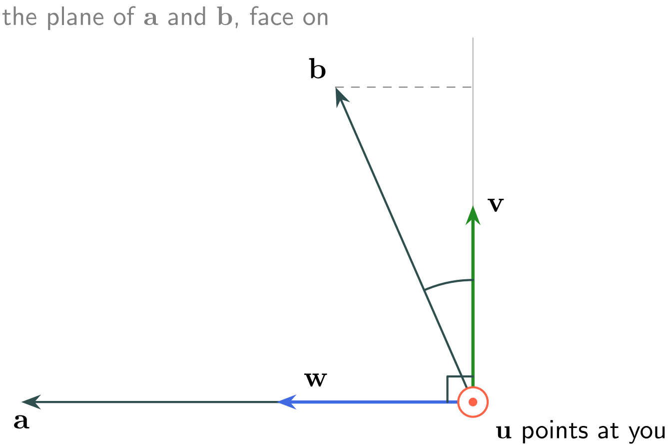
$\mathbf{v}$ is $\mathbf{b}$, straightened up

the plane of $\mathbf{a}$ and $\mathbf{b}$, face on



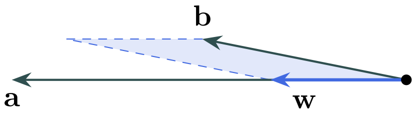
- $\mathbf{w}$ stays exactly on $\mathbf{a}$
- $\mathbf{v}$: as near $\mathbf{b}$ as allowed

$\mathbf{v}$ is $\mathbf{b}$, straightened up



- $\mathbf{w}$ stays exactly on $\mathbf{a}$
- $\mathbf{v}$: as near $\mathbf{b}$ as allowed
- Still perpendicular to $\mathbf{w}$

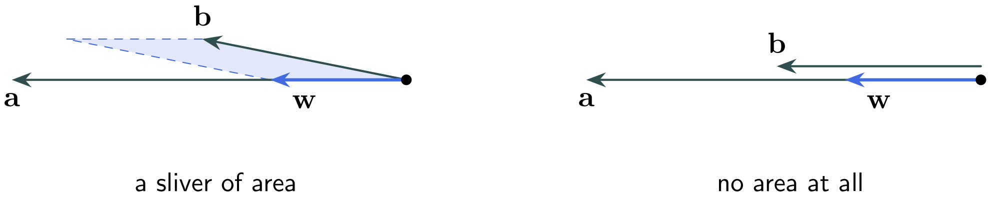
Parallel arrows: the recipe breaks



a sliver of area

- Nearly parallel: a shaky $\mathbf{u}$

Parallel arrows: the recipe breaks

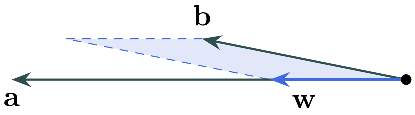


- Nearly parallel: a shaky $\mathbf{u}$
- Parallel: $\mathbf{b} \times \mathbf{w} = \mathbf{0}$

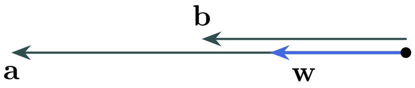
Parallel arrows: the recipe breaks

$$\mathbf{u} = \frac{\mathbf{b} \times \mathbf{w}}{\|\mathbf{b} \times \mathbf{w}\|}$$

this is 0



a sliver of area

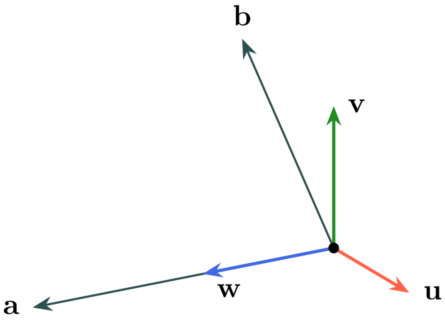


no area at all

- Nearly parallel: a shaky $\mathbf{u}$

- Parallel: $\mathbf{b} \times \mathbf{w} = \mathbf{0}$
- Dividing by zero: no frame

Recap: a frame from two arrows



$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

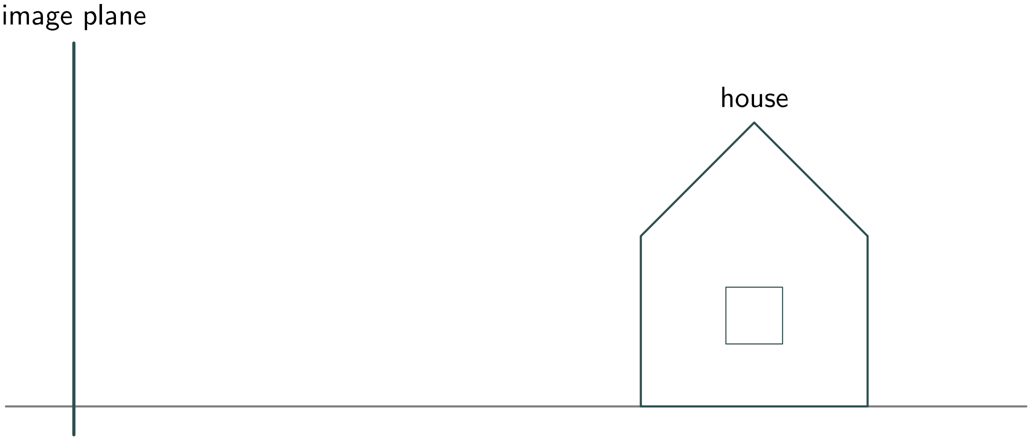
$$\mathbf{u} = \frac{\mathbf{b} \times \mathbf{w}}{\|\mathbf{b} \times \mathbf{w}\|}$$

$$\mathbf{v} = \mathbf{w} \times \mathbf{u}$$

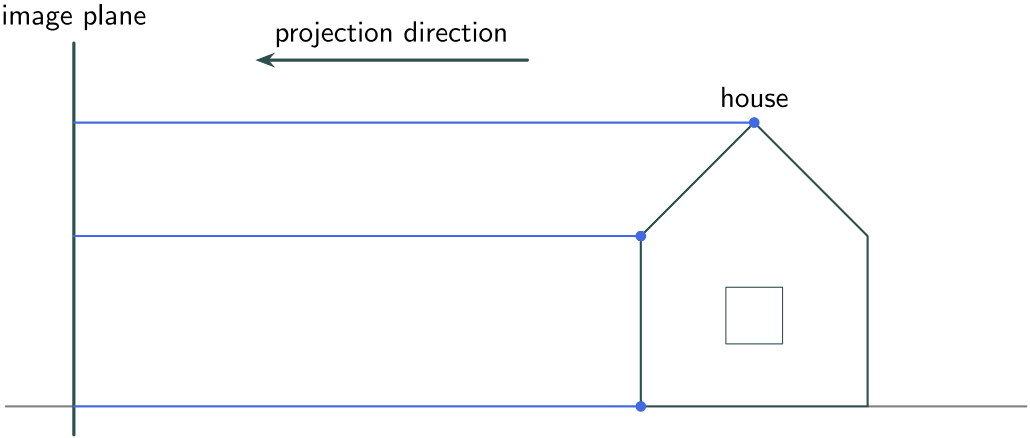
The camera

A frame, a window, many rays

Parallel projection: every line shares one direction

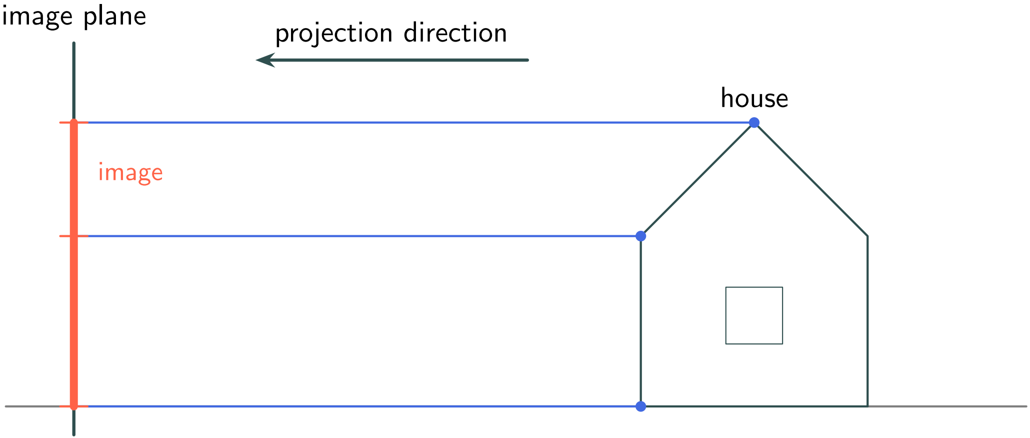


Parallel projection: every line shares one direction



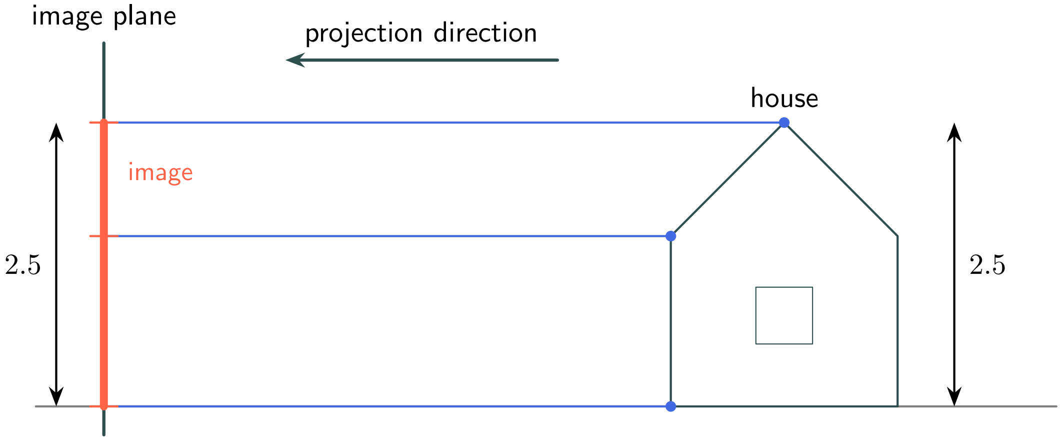
- Slide along one direction

Parallel projection: every line shares one direction



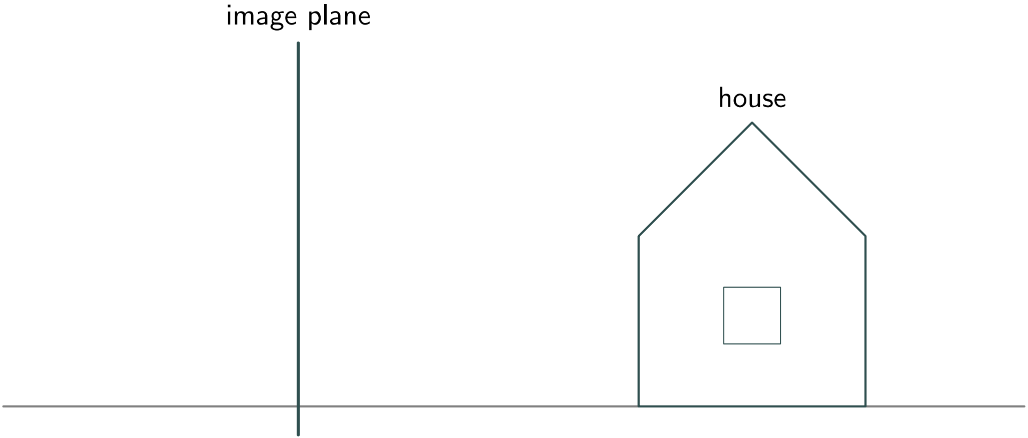
- Slide along one direction
- Stop at the image plane

Parallel projection: every line shares one direction

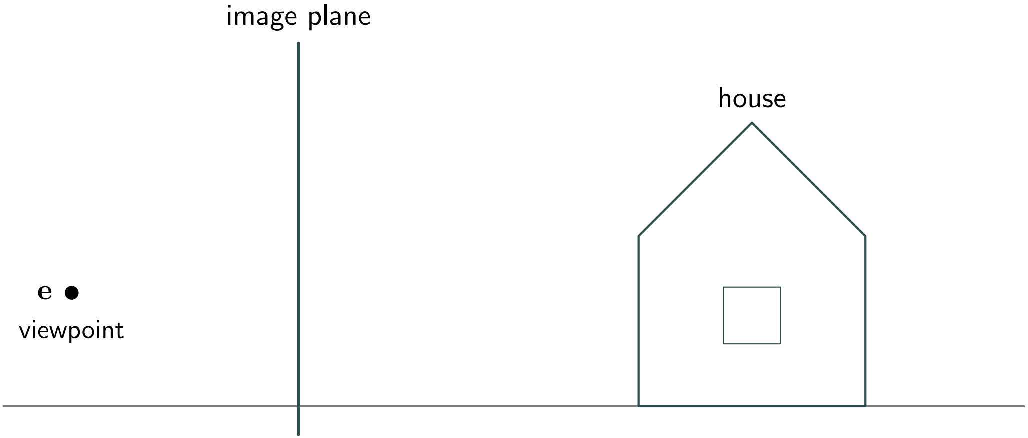


- Slide along one direction
- Stop at the image plane
- Image: same height

Perspective: every line passes through one point

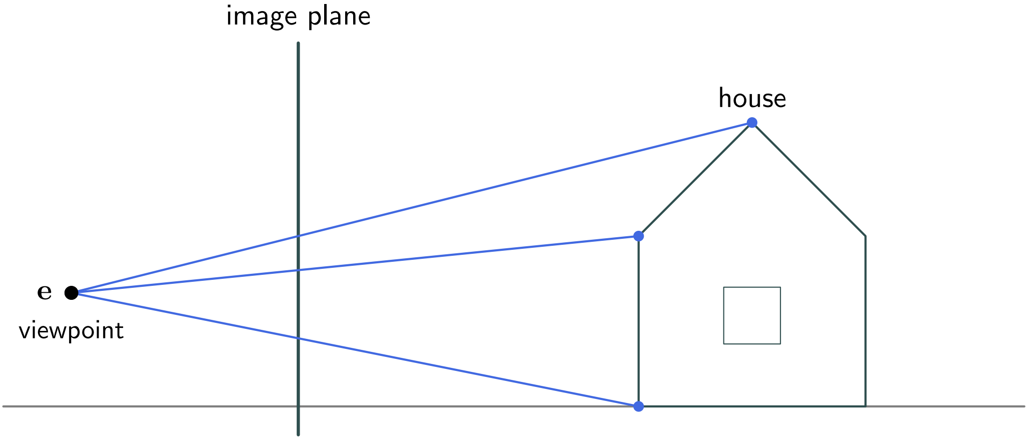


Perspective: every line passes through one point



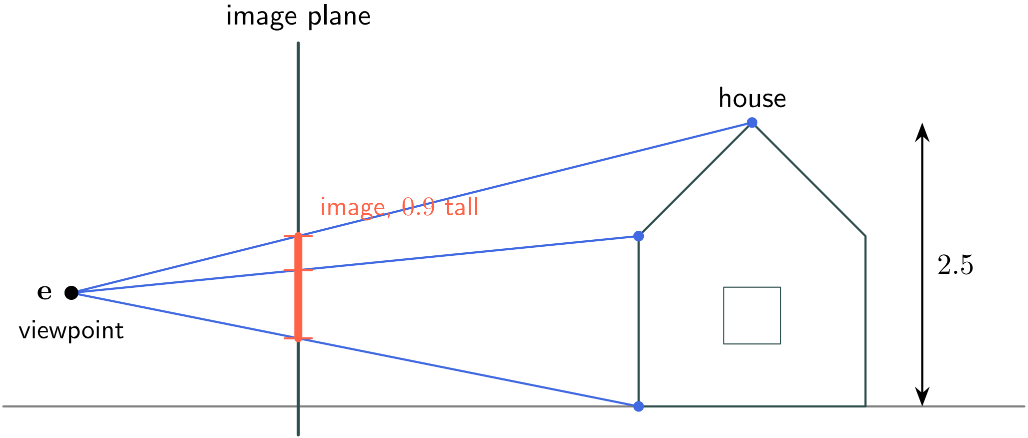
- A single viewpoint

Perspective: every line passes through one point



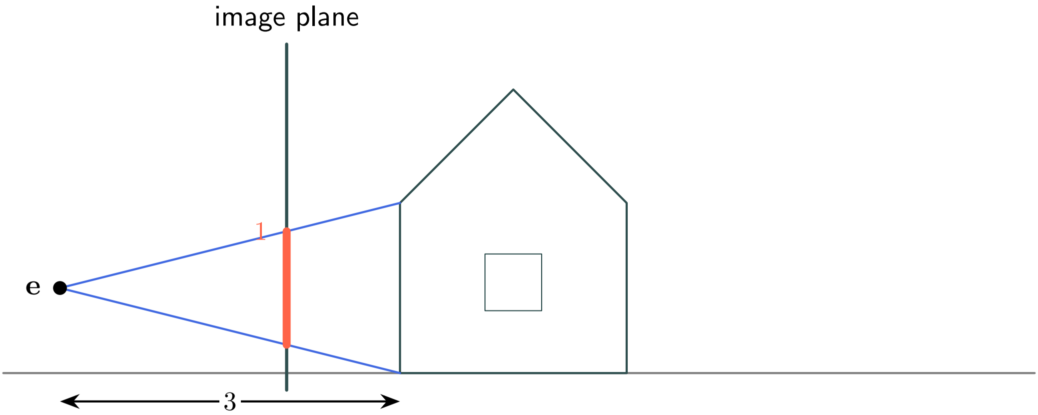
- A single viewpoint
- All lines pass through it

Perspective: every line passes through one point

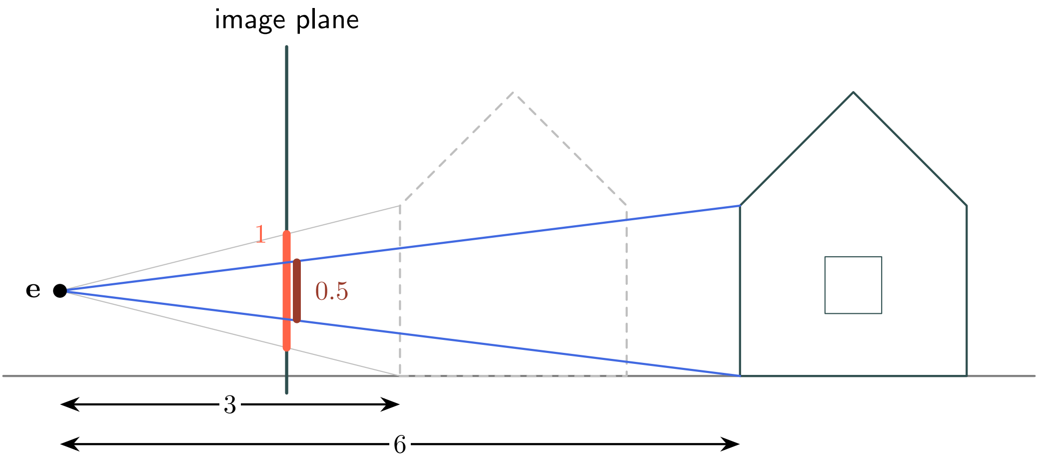


- A single viewpoint
- All lines pass through it
- Drawn at the crossing

Twice as far, half as tall

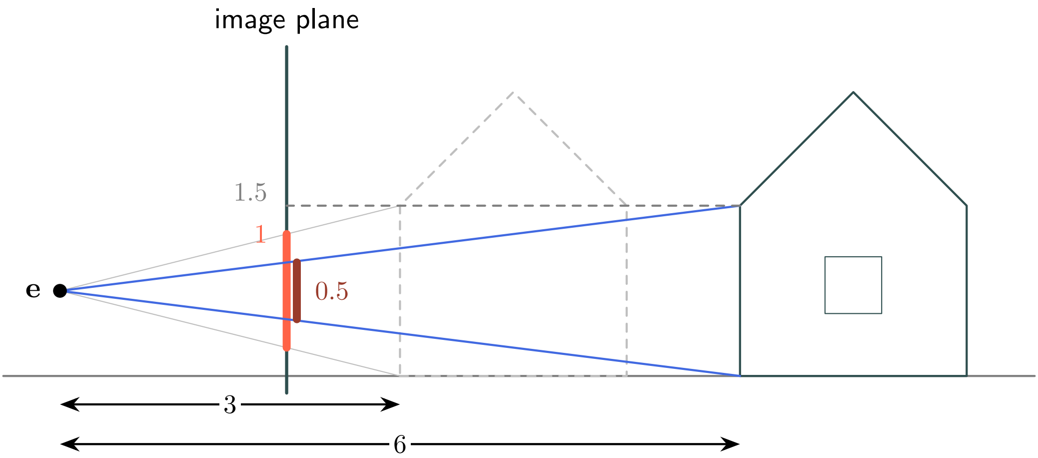


Twice as far, half as tall



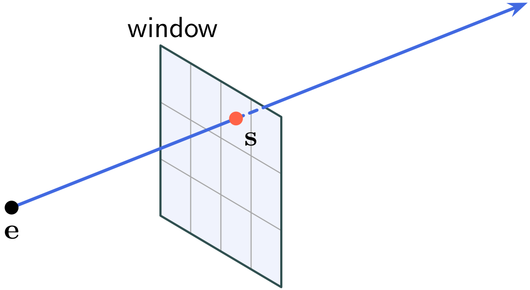
- Same house, twice as far
- Its image: half as tall

Twice as far, half as tall



- Same house, twice as far
- Its image: half as tall
- Parallel: no shrinking

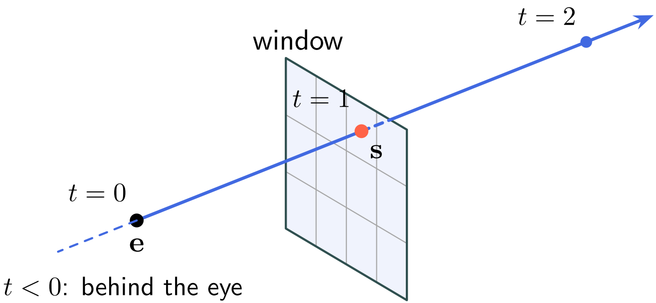
A viewing ray needs $\mathbf{e}$ and $\mathbf{s}$



- Lecture 10's ray, through $\mathbf{s}$

  $$\mathbf{p}(t) = \mathbf{e} + t\,(\mathbf{s} - \mathbf{e})$$

A viewing ray needs $\mathbf{e}$ and $\mathbf{s}$

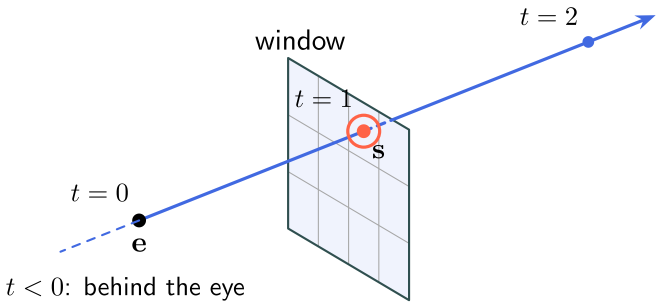


- Lecture 10's ray, through $\mathbf{s}$

  $\mathbf{p}(t) = \mathbf{e} + t\,(\mathbf{s} - \mathbf{e})$

- $t = 1$ lands on $\mathbf{s}$

A viewing ray needs $\mathbf{e}$ and $\mathbf{s}$

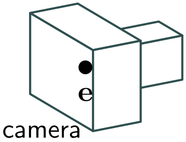


- Lecture 10's ray, through $\mathbf{s}$

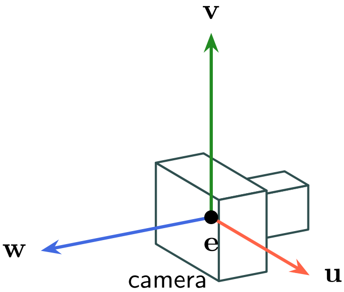
  $\mathbf{p}(t) = \mathbf{e} + t\,(\mathbf{s} - \mathbf{e})$

- $t = 1$ lands on $\mathbf{s}$
- $\mathbf{e}$ is given; find $\mathbf{s}$

The camera frame: right, up, back

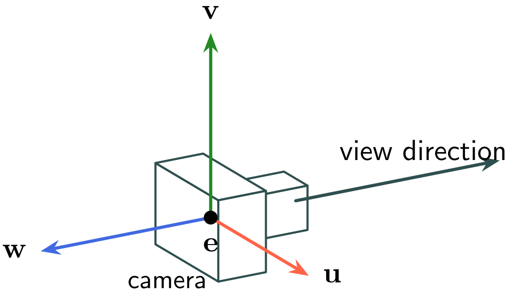


The camera frame: right, up, back



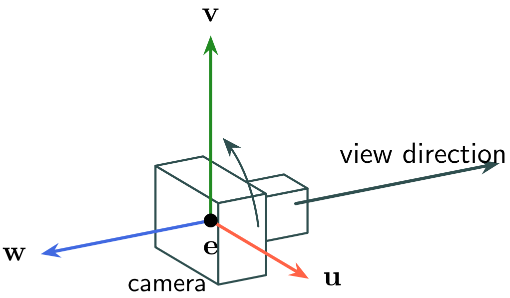
- $\mathbf{u}$ right, $\mathbf{v}$ up, $\mathbf{w}$ back

The camera frame: right, up, back



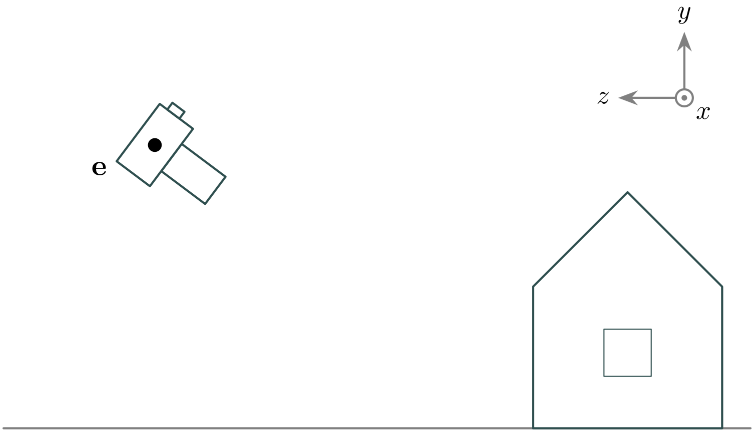
- $\mathbf{u}$ right, $\mathbf{v}$ up, $\mathbf{w}$ back
- It looks along $-\mathbf{w}$

The camera frame: right, up, back

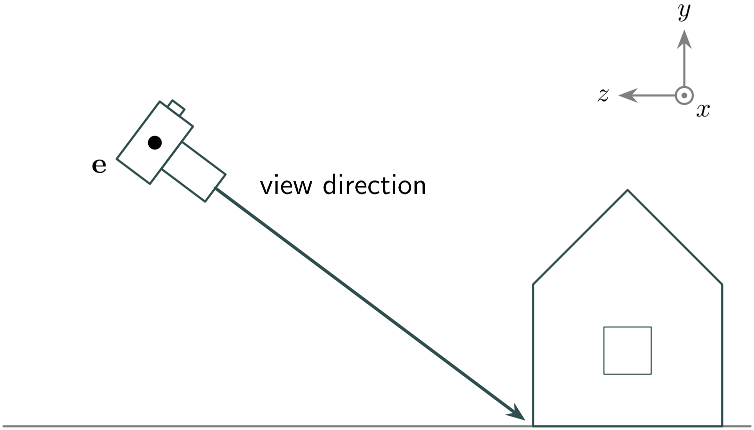


- $\mathbf{u}$ right, $\mathbf{v}$ up, $\mathbf{w}$ back
- It looks along $-\mathbf{w}$
- Right-handed: $\mathbf{w} = \mathbf{u} \times \mathbf{v}$

Two inputs aim the camera

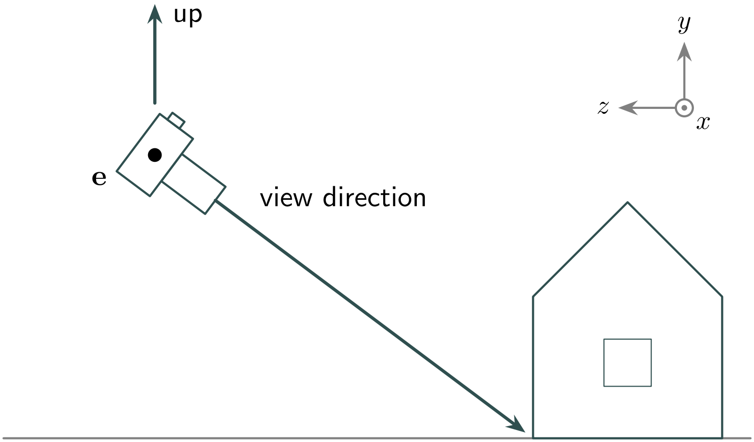


Two inputs aim the camera



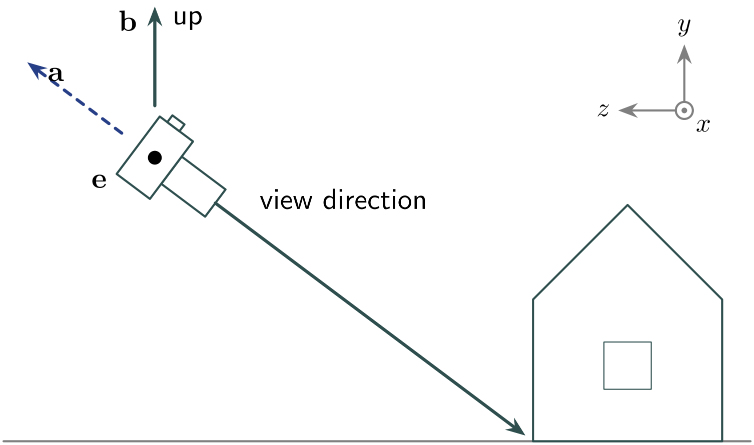
- View direction: where it looks

Two inputs aim the camera



- View direction: where it looks
- Up: which way is the top

Two inputs aim the camera

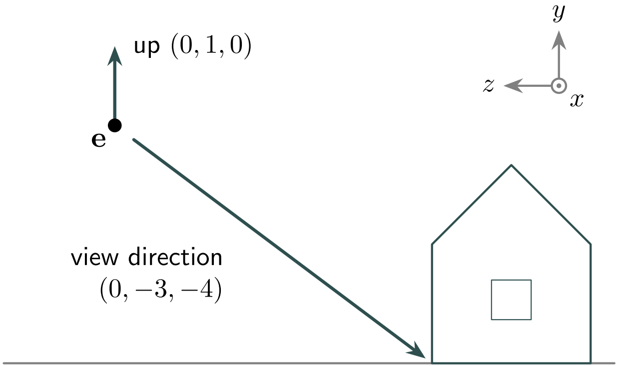


- View direction: where it looks
- Up: which way is the top
- Then the two-arrow recipe

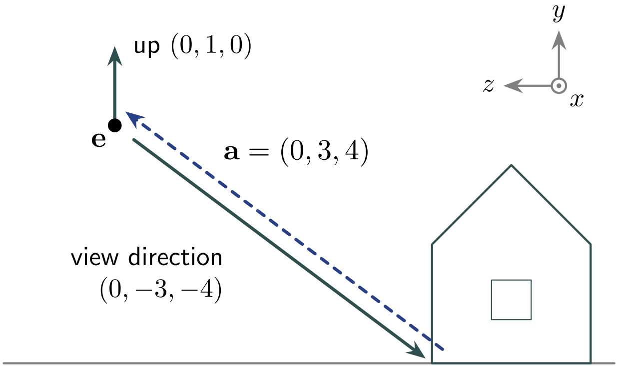
$\mathbf{a} = -(\text{view direction})$

$\mathbf{b} = \text{up}$

Worked: $\mathbf{w}$ points back along the view



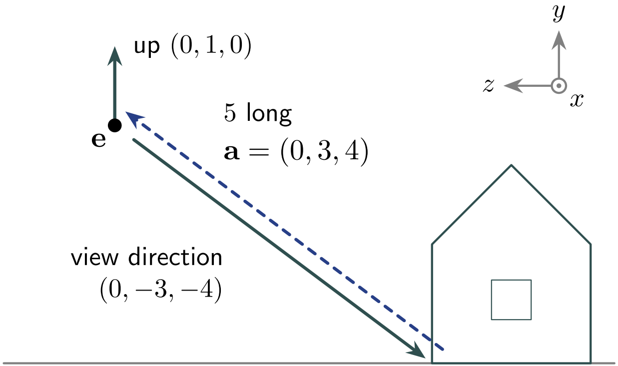
Worked: $\mathbf{w}$ points back along the view



$$\mathbf{a} = -(0, -3, -4) = (0,\ 3,\ 4)$$

flip the view direction

Worked: $\mathbf{w}$ points back along the view



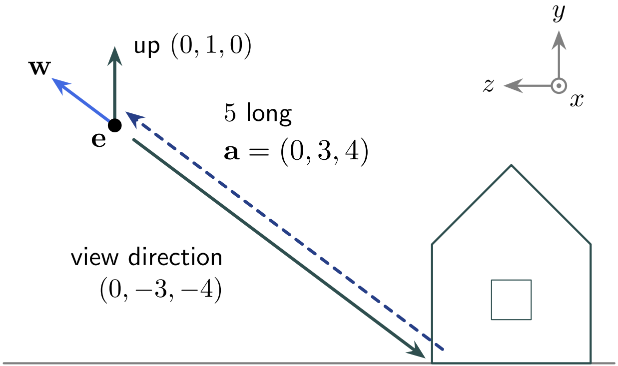
$$\mathbf{a} = -(0,-3,-4) = (0,\ 3,\ 4)$$

flip the view direction

$$\|\mathbf{a}\| = \sqrt{0^2+3^2+4^2} = \sqrt{25} = 5$$

its length

Worked: $\mathbf{w}$ points back along the view



$$\mathbf{a} = -(0, -3, -4) = (0,\ 3,\ 4)$$

flip the view direction

$$\|\mathbf{a}\| = \sqrt{0^2 + 3^2 + 4^2} = \sqrt{25} = 5$$

its length

$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|} = \left(\frac{0}{5},\ \frac{3}{5},\ \frac{4}{5}\right)$$

$$= (0,\ 0.6,\ 0.8)$$

divide each number

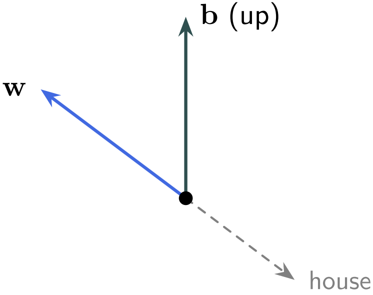
- Away from the scene
- Exactly one unit long

Worked: $\mathbf{u}$ from $\mathbf{b} \times \mathbf{w}$

side view, close to $\mathbf{e}$

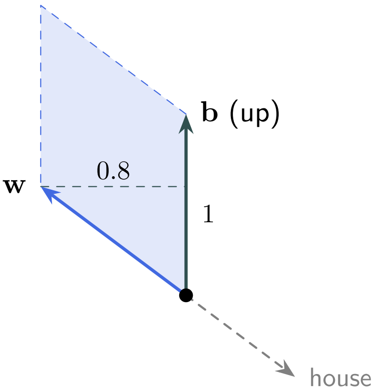
$$\mathbf{b} = (0,\ 1,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$$

slots: $\left(y_b z_w - z_b y_w,\ \ z_b x_w - x_b z_w,\ \ x_b y_w - y_b x_w \right)$



Worked: $\mathbf{u}$ from $\mathbf{b} \times \mathbf{w}$

side view, close to $\mathbf{e}$



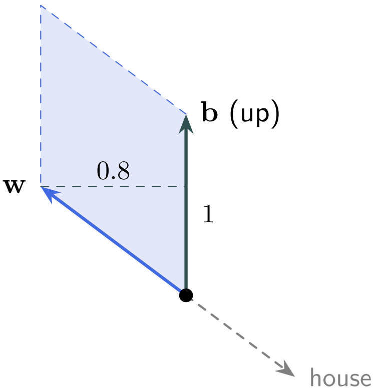
$$\mathbf{b} = (0,\ 1,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$$

slots: $(\ y_b z_w - z_b y_w, \quad z_b x_w - x_b z_w, \quad x_b y_w - y_b x_w\ )$

x slot: $(1)(0.8) - (0)(0.6) = 0.8 - 0 = 0.8$

Worked: $\mathbf{u}$ from $\mathbf{b} \times \mathbf{w}$

side view, close to $\mathbf{e}$



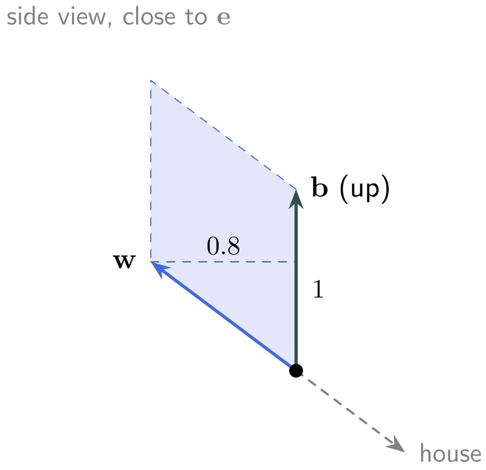
$$\mathbf{b} = (0,\ 1,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$$

slots: $(\, y_b z_w - z_b y_w, \;\; z_b x_w - x_b z_w, \;\; x_b y_w - y_b x_w \,)$

x slot: $(1)(0.8) - (0)(0.6) = 0.8 - 0 = 0.8$

y slot: $(0)(0) - (0)(0.8) = 0 - 0 = 0$

Worked: $\mathbf{u}$ from $\mathbf{b} \times \mathbf{w}$



$$\mathbf{b} = (0,\ 1,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$$

slots: $\left(y_b z_w - z_b y_w,\ \ z_b x_w - x_b z_w,\ \ x_b y_w - y_b x_w \right)$

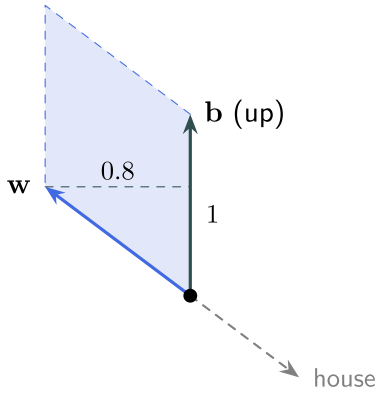
x slot: $(1)(0.8) - (0)(0.6) = 0.8 - 0 = 0.8$

y slot: $(0)(0) - (0)(0.8) = 0 - 0 = 0$

z slot: $(0)(0.6) - (1)(0) = 0 - 0 = 0$

Worked: $\mathbf{u}$ from $\mathbf{b} \times \mathbf{w}$

side view, close to $\mathbf{e}$



$\mathbf{b} = (0,\ 1,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$

slots: $(\ y_b z_w - z_b y_w,\ \ z_b x_w - x_b z_w,\ \ x_b y_w - y_b x_w\ )$

x slot: $(1)(0.8) - (0)(0.6) = 0.8 - 0 = 0.8$

y slot: $(0)(0) - (0)(0.8) = 0 - 0 = 0$

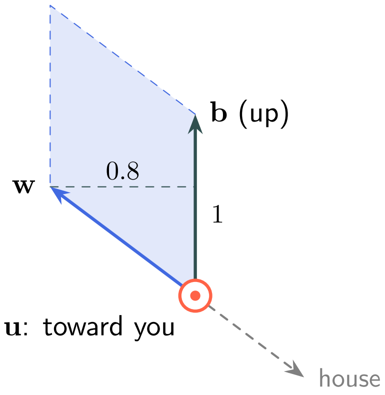
z slot: $(0)(0.6) - (1)(0) = 0 - 0 = 0$

$\mathbf{b} \times \mathbf{w} = (0.8,\ 0,\ 0), \quad$ length 0.8

- Length 0.8: the shaded area

Worked: $\mathbf{u}$ from $\mathbf{b} \times \mathbf{w}$

side view, close to $\mathbf{e}$



$$\mathbf{b} = (0,\ 1,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$$

slots: $(\ y_b z_w - z_b y_w,\quad z_b x_w - x_b z_w,\quad x_b y_w - y_b x_w\ )$

x slot: $(1)(0.8) - (0)(0.6) = 0.8 - 0 = 0.8$

y slot: $(0)(0) - (0)(0.8) = 0 - 0 = 0$

z slot: $(0)(0.6) - (1)(0) = 0 - 0 = 0$

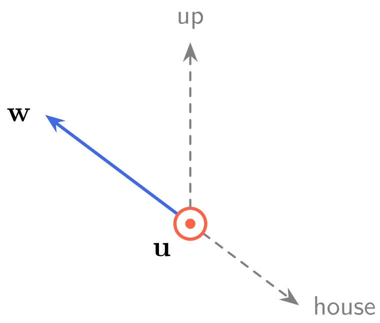
$\mathbf{b} \times \mathbf{w} = (0.8,\ 0,\ 0)$, length 0.8

$$\mathbf{u} = \frac{(0.8,\ 0,\ 0)}{0.8} = (1,\ 0,\ 0)$$

- Length 0.8: the shaded area
- $\mathbf{u}$ points at you, along x

Worked: $\mathbf{v}$, and it is not up

side view, close to $\mathbf{e}$

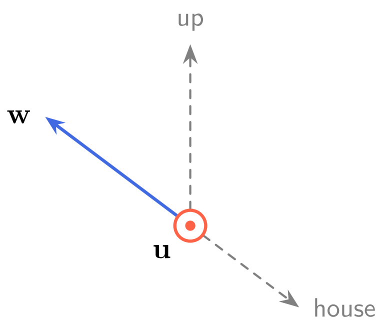


$$\mathbf{w} = (0,\ 0.6,\ 0.8), \quad \mathbf{u} = (1,\ 0,\ 0)$$

slots: $(\, y_w z_u - z_w y_u,\;\; z_w x_u - x_w z_u,\;\; x_w y_u - y_w x_u \,)$

Worked: $\mathbf{v}$, and it is not up

side view, close to $\mathbf{e}$



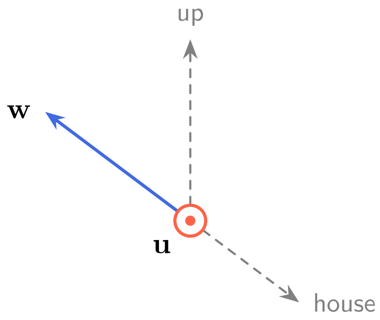
$$\mathbf{w} = (0,\ 0.6,\ 0.8), \quad \mathbf{u} = (1,\ 0,\ 0)$$

slots: $(\, y_w z_u - z_w y_u,\ \ z_w x_u - x_w z_u,\ \ x_w y_u - y_w x_u \,)$

x slot: $(0.6)(0) - (0.8)(0) = 0 - 0 = 0$

Worked: $\mathbf{v}$, and it is not up

side view, close to $\mathbf{e}$



$$\mathbf{w} = (0,\ 0.6,\ 0.8), \quad \mathbf{u} = (1,\ 0,\ 0)$$

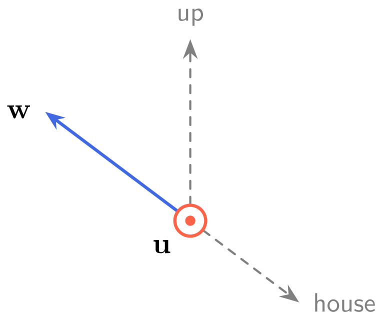
slots: $(\, y_w z_u - z_w y_u,\ \ z_w x_u - x_w z_u,\ \ x_w y_u - y_w x_u \,)$

x slot: $(0.6)(0) - (0.8)(0) = 0 - 0 = 0$

y slot: $(0.8)(1) - (0)(0) = 0.8 - 0 = 0.8$

Worked: $\mathbf{v}$, and it is not up

side view, close to $\mathbf{e}$



$$\mathbf{w} = (0,\ 0.6,\ 0.8), \quad \mathbf{u} = (1,\ 0,\ 0)$$

slots: $(\, y_w z_u - z_w y_u, \;\; z_w x_u - x_w z_u, \;\; x_w y_u - y_w x_u \,)$

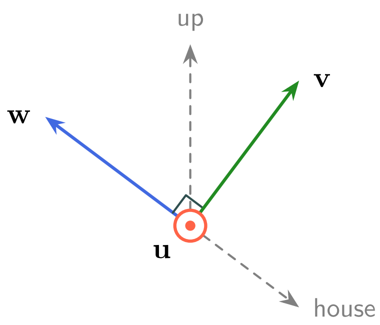
x slot: $(0.6)(0) - (0.8)(0) = 0 - 0 = 0$

y slot: $(0.8)(1) - (0)(0) = 0.8 - 0 = 0.8$

z slot: $(0)(0) - (0.6)(1) = 0 - 0.6 = -0.6$

Worked: $\mathbf{v}$, and it is not up

side view, close to $\mathbf{e}$



$\mathbf{w} = (0,\ 0.6,\ 0.8), \quad \mathbf{u} = (1,\ 0,\ 0)$

slots: $(\ y_w z_u - z_w y_u,\ \ z_w x_u - x_w z_u,\ \ x_w y_u - y_w x_u\ )$

x slot: $(0.6)(0) - (0.8)(0) = 0 - 0 = 0$

y slot: $(0.8)(1) - (0)(0) = 0.8 - 0 = 0.8$

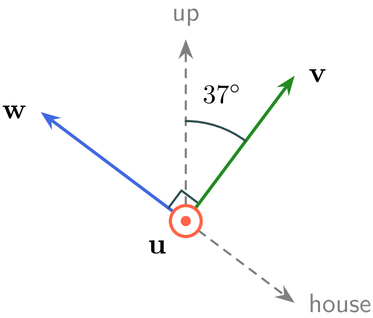
z slot: $(0)(0) - (0.6)(1) = 0 - 0.6 = -0.6$

$\mathbf{v} = \mathbf{w} \times \mathbf{u} = (0,\ 0.8,\ -0.6)$

- $\mathbf{v}$ tips forward with the camera

Worked: $\mathbf{v}$, and it is not up

side view, close to $\mathbf{e}$



$\mathbf{w} = (0,\ 0.6,\ 0.8), \quad \mathbf{u} = (1,\ 0,\ 0)$

slots: $(\, y_w z_u - z_w y_u, \;\; z_w x_u - x_w z_u, \;\; x_w y_u - y_w x_u \,)$

x slot: $(0.6)(0) - (0.8)(0) = 0 - 0 = 0$

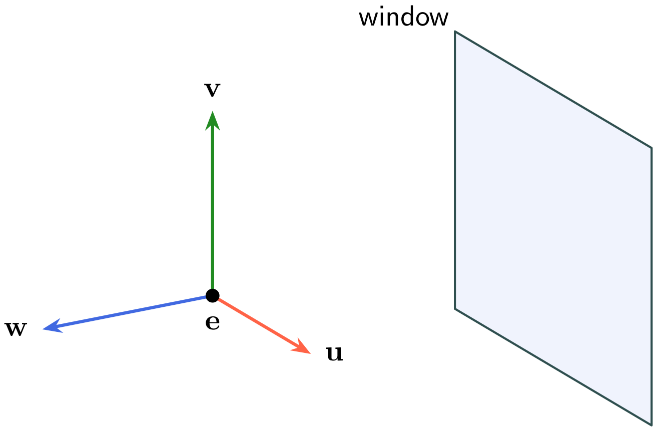
y slot: $(0.8)(1) - (0)(0) = 0.8 - 0 = 0.8$

z slot: $(0)(0) - (0.6)(1) = 0 - 0.6 = -0.6$

$\mathbf{v} = \mathbf{w} \times \mathbf{u} = (0,\ 0.8,\ -0.6)$

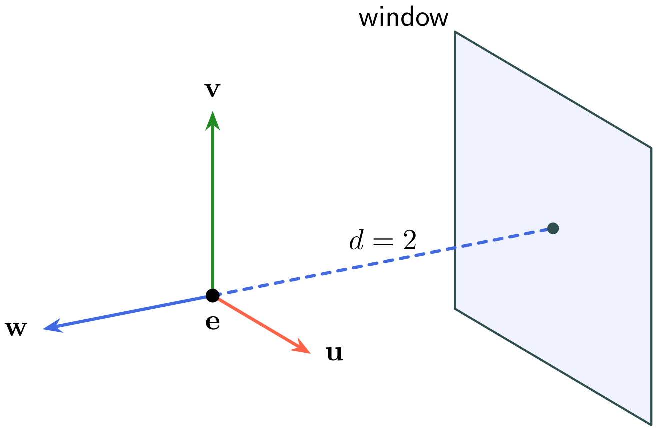
- $\mathbf{v}$ tips forward with the camera
- $\mathbf{v}$ is not up
- Level camera: $\mathbf{v}$ equals up

Goal: place the window the rays pass through



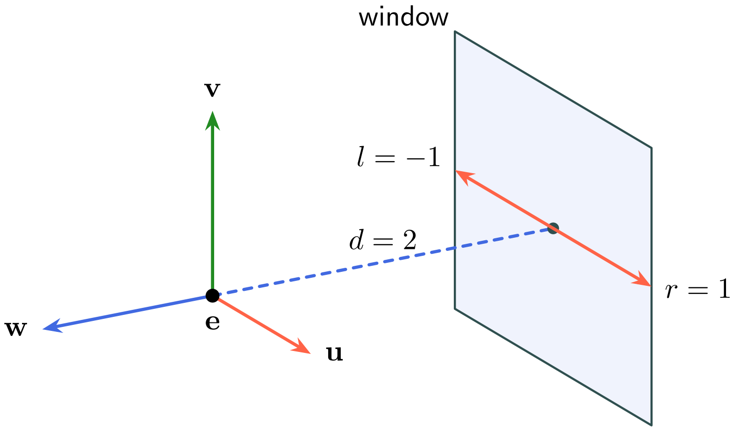
- Window: the rectangle that becomes the picture

Goal: place the window the rays pass through



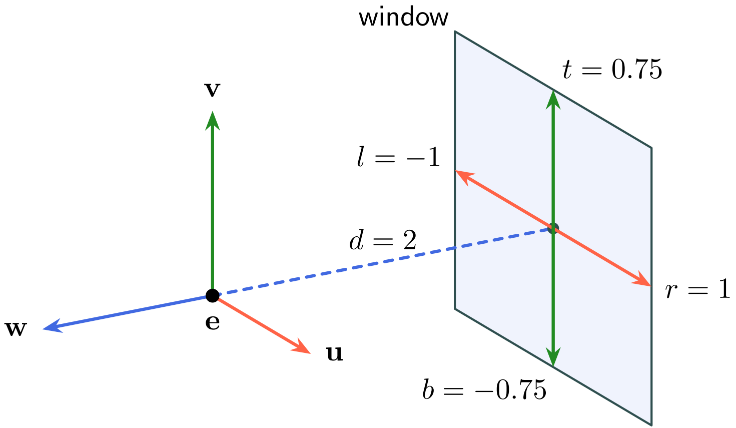
- Window: the rectangle that becomes the picture
- d ahead of $\mathbf{e}$, along $-\mathbf{w}$

Goal: place the window the rays pass through



- Window: the rectangle that becomes the picture
- d ahead of $\mathbf{e}$, along $-\mathbf{w}$
- l, r: left, right, along $\mathbf{u}$

Goal: place the window the rays pass through

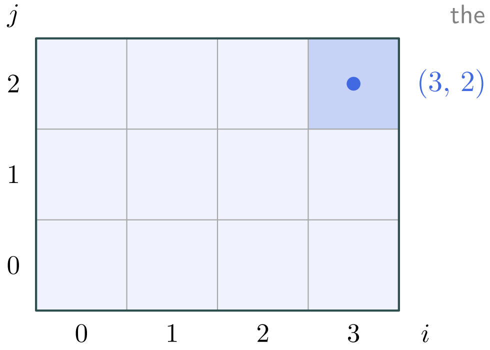


- Window: the rectangle that becomes the picture
- d ahead of $\mathbf{e}$, along $-\mathbf{w}$
- l, r: left, right, along $\mathbf{u}$
- b, t: bottom, top, along $\mathbf{v}$

One spot, two kinds of coordinates

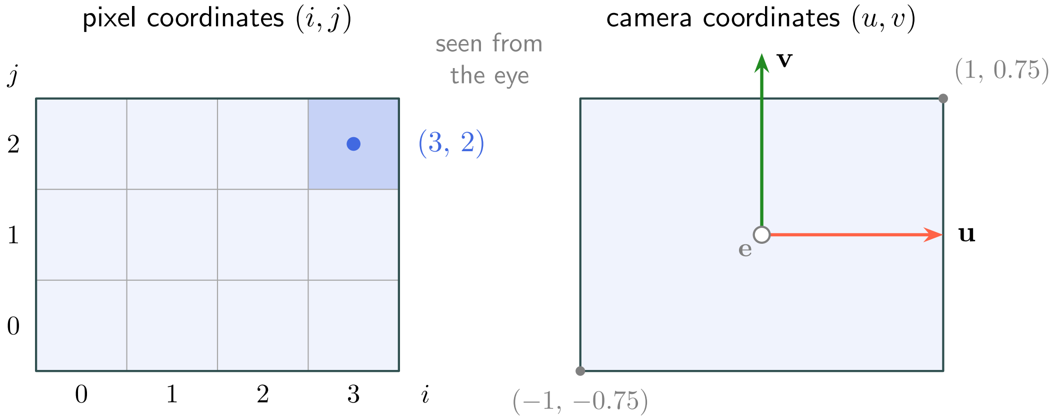
pixel coordinates (i, j)

seen from the eye



- Counts cells, from the bottom-left
- Whole numbers; the image uses them

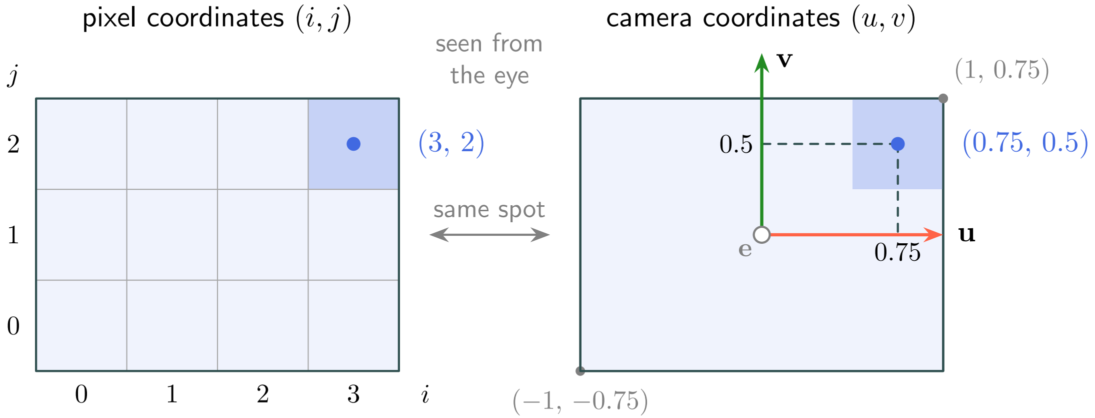
One spot, two kinds of coordinates



- Counts cells, from the bottom-left
- Whole numbers; the image uses them

- Measures from $\mathbf{e}$, along $\mathbf{u}$ and $\mathbf{v}$

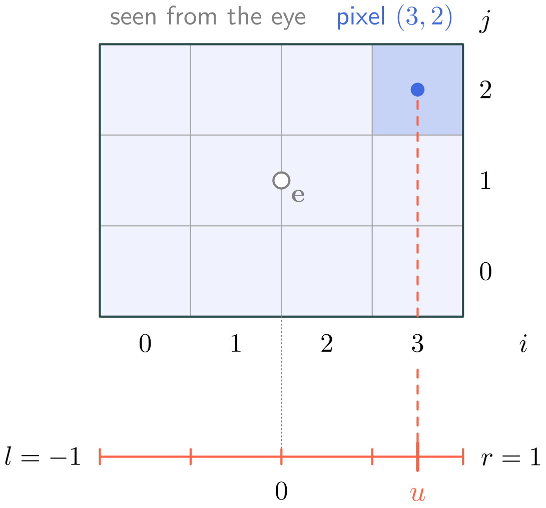
One spot, two kinds of coordinates



- Counts cells, from the bottom-left
- Whole numbers; the image uses them

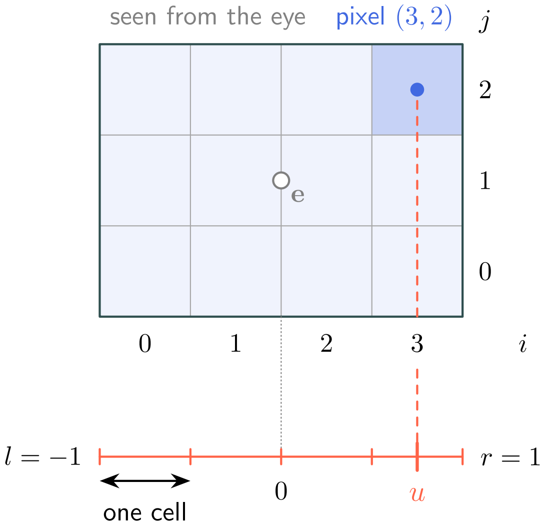
- Measures from $\mathbf{e}$, along $\mathbf{u}$ and $\mathbf{v}$
- Real numbers; the ray uses them

Goal: from pixel (i, j) to position (u, v)



- Ray aims at the pixel's centre
- u: how far right of $\mathbf{e}$

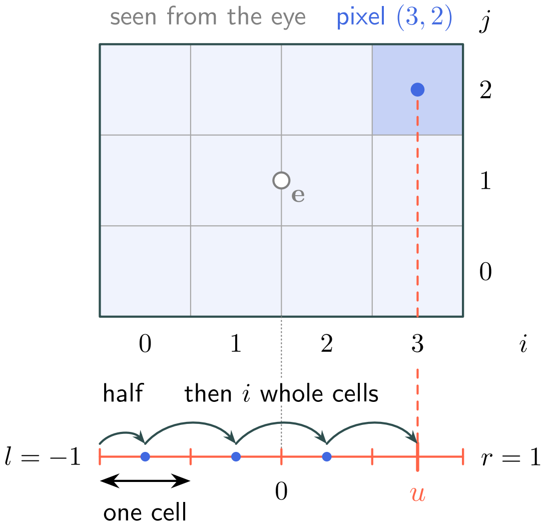
Goal: from pixel (i, j) to position (u, v)



- Ray aims at the pixel's centre
- u: how far right of $\mathbf{e}$

  cell width $= (r - l)/n_x$

Goal: from pixel (i, j) to position (u, v)



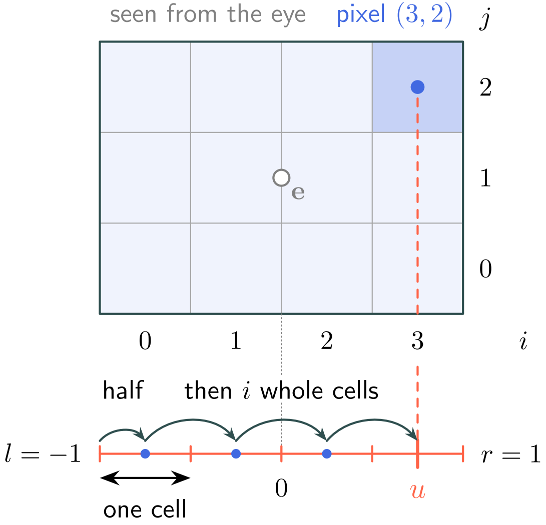
- Ray aims at the pixel's centre
- u: how far right of $\mathbf{e}$

  cell width $= (r - l)/n_x$

- Half a cell, then i cells

  $u = l + (0.5 + i) \times$ cell width

Goal: from pixel (i, j) to position (u, v)



- Ray aims at the pixel's centre
- u: how far right of $\mathbf{e}$

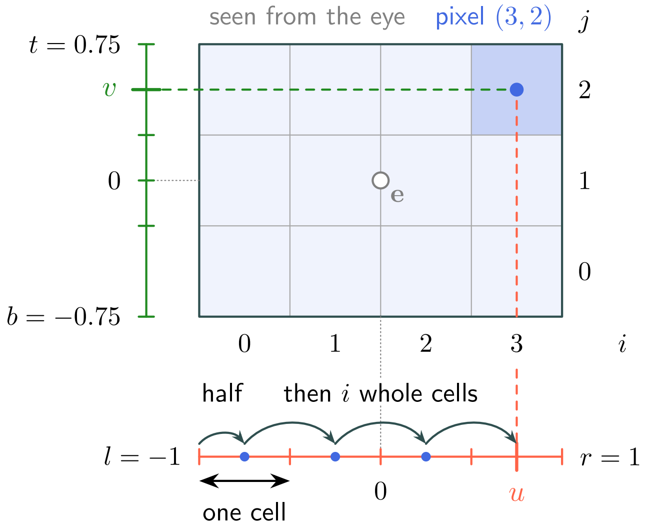
  cell width $= (r - l)/n_x$

- Half a cell, then i cells

  $u = l + (0.5 + i) \times$ cell width

  $$u = l + (r - l)\,\frac{i + 0.5}{n_x}$$

Goal: from pixel (i, j) to position (u, v)



- Ray aims at the pixel's centre
- u: how far right of $\mathbf{e}$

  cell width $= (r - l)/n_x$

- Half a cell, then i cells

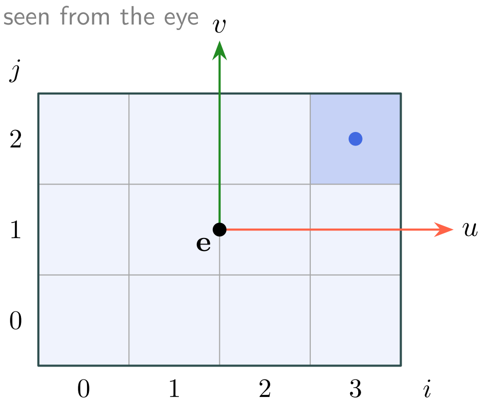
  $$u = l + (0.5 + i) \times \text{ cell width}$$

  $$u = l + (r - l)\,\frac{i + 0.5}{n_x}$$

- v: same walk, up from b

  $$v = b + (t - b)\,\frac{j + 0.5}{n_y}$$

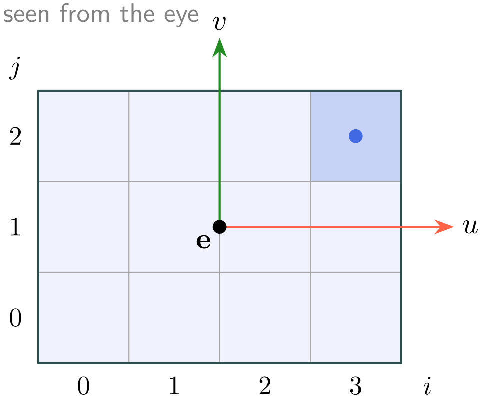
Worked: pixel $(3, 2)$ sits at $(0.75,\ 0.5)$



$$l = -1, \quad r = 1, \quad n_x = 4, \quad i = 3$$

- Top-right pixel: $i = 3$, $j = 2$

Worked: pixel $(3, 2)$ sits at $(0.75,\ 0.5)$

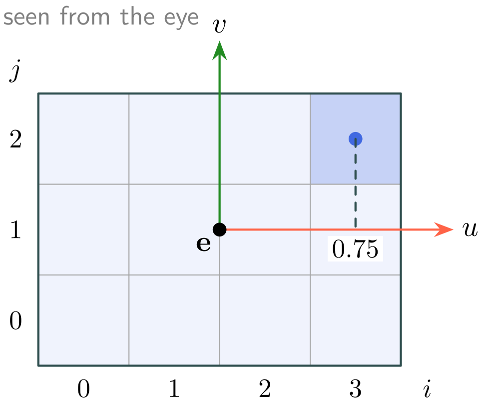


- Top-right pixel: $i = 3,\ j = 2$

$$l = -1, \quad r = 1, \quad n_x = 4, \quad i = 3$$

$$u = -1 + \frac{(1 - (-1))\,(3 + 0.5)}{4}$$

Worked: pixel $(3, 2)$ sits at $(0.75,\ 0.5)$



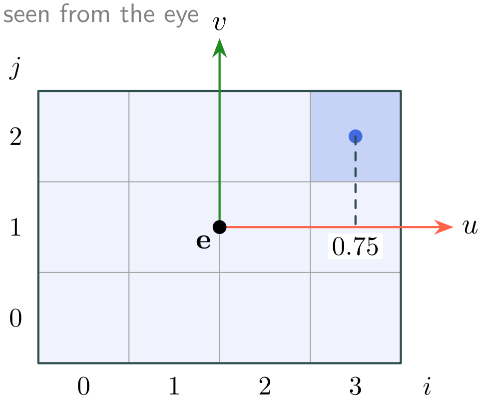
- Top-right pixel: $i = 3,\ j = 2$

$$l = -1, \quad r = 1, \quad n_x = 4, \quad i = 3$$

$$u = -1 + \frac{(1 - (-1))\,(3 + 0.5)}{4}$$

$$= -1 + \frac{2 \times 3.5}{4} = -1 + 1.75 = 0.75$$

Worked: pixel $(3, 2)$ sits at $(0.75,\ 0.5)$



$$l = -1, \quad r = 1, \quad n_x = 4, \quad i = 3$$

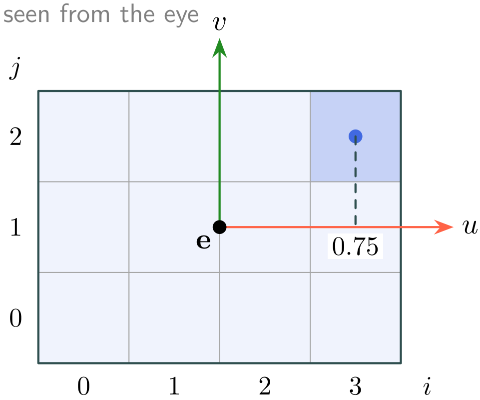
$$u = -1 + \frac{(1 - (-1))\,(3 + 0.5)}{4}$$

$$= -1 + \frac{2 \times 3.5}{4} = -1 + 1.75 = 0.75$$

$$b = -0.75, \quad t = 0.75, \quad n_y = 3, \quad j = 2$$

- Top-right pixel: $i = 3,\ j = 2$

Worked: pixel $(3, 2)$ sits at $(0.75,\ 0.5)$



- Top-right pixel: $i = 3,\ j = 2$

$$l = -1, \quad r = 1, \quad n_x = 4, \quad i = 3$$

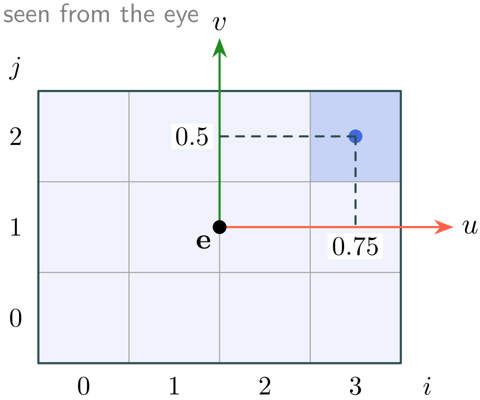
$$u = -1 + \frac{(1 - (-1))\,(3 + 0.5)}{4}$$

$$= -1 + \frac{2 \times 3.5}{4} = -1 + 1.75 = 0.75$$

$$b = -0.75, \quad t = 0.75, \quad n_y = 3, \quad j = 2$$

$$v = -0.75 + \frac{(0.75 - (-0.75))\,(2 + 0.5)}{3}$$

Worked: pixel $(3, 2)$ sits at $(0.75,\ 0.5)$



- Top-right pixel: $i = 3$, $j = 2$
- Its centre: right 0.75, up 0.5

$$l = -1, \quad r = 1, \quad n_x = 4, \quad i = 3$$

$$u = -1 + \frac{(1 - (-1))\,(3 + 0.5)}{4}$$

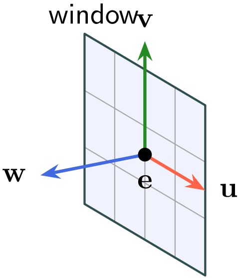
$$= -1 + \frac{2 \times 3.5}{4} = -1 + 1.75 = 0.75$$

$$b = -0.75, \quad t = 0.75, \quad n_y = 3, \quad j = 2$$

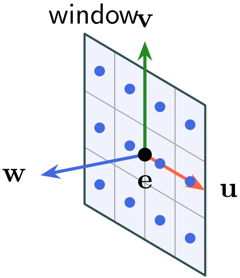
$$v = -0.75 + \frac{(0.75 - (-0.75))\,(2 + 0.5)}{3}$$

$$= -0.75 + \frac{1.5 \times 2.5}{3} = -0.75 + 1.25 = 0.5$$

Orthographic: same direction, different origins



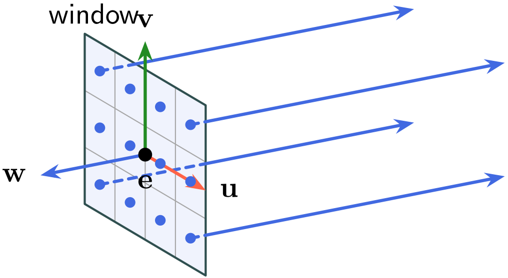
Orthographic: same direction, different origins



- Every ray starts on its pixel

  origin $= \mathbf{e} + u\,\mathbf{u} + v\,\mathbf{v}$

Orthographic: same direction, different origins



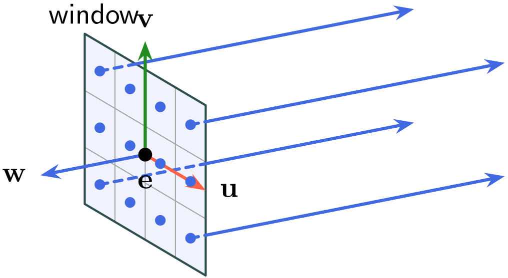
- Every ray starts on its pixel

  origin $= \mathbf{e} + u\,\mathbf{u} + v\,\mathbf{v}$

- Every ray points along $-\mathbf{w}$

  direction $= -\mathbf{w}$

Orthographic: same direction, different origins



- Every ray starts on its pixel

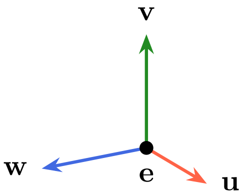
  origin $= \mathbf{e} + u\,\mathbf{u} + v\,\mathbf{v}$

- Every ray points along $-\mathbf{w}$

  direction $= -\mathbf{w}$

- A parallel projection

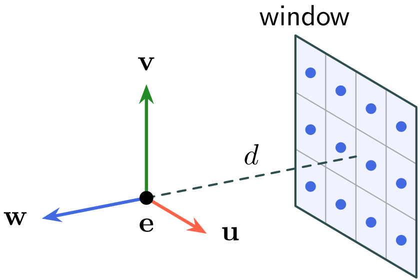
Perspective: same origin, different directions



- Every ray starts at $\mathbf{e}$

  origin $= \mathbf{e}$

Perspective: same origin, different directions

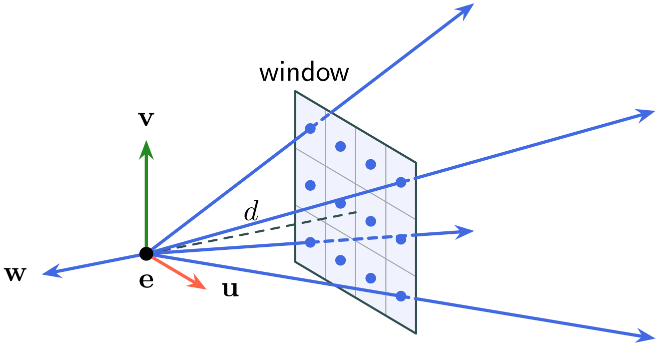


- Every ray starts at $\mathbf{e}$

  origin $= \mathbf{e}$

- The window sits d in front

Perspective: same origin, different directions



- Every ray starts at $\mathbf{e}$

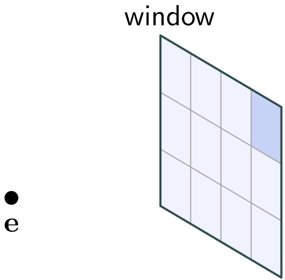
  origin $= \mathbf{e}$

- The window sits d in front

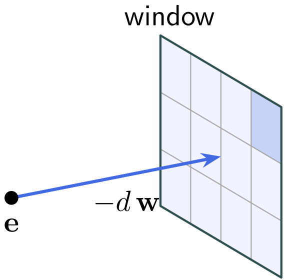
- Every pixel: its own direction

  direction $= -d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$

Walk from $\mathbf{e}$ to $\mathbf{s}$ in three legs

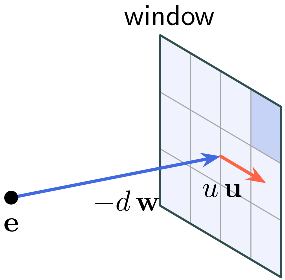


Walk from e to s in three legs



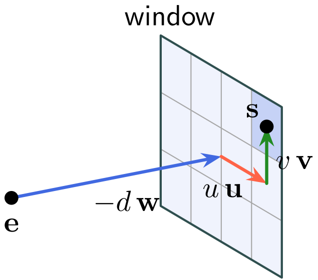
- Forward d, along $-\mathbf{w}$

Walk from $\mathbf{e}$ to $\mathbf{s}$ in three legs



- Forward d, along $-\mathbf{w}$
- Across u, then up v

Walk from $\mathbf{e}$ to $\mathbf{s}$ in three legs

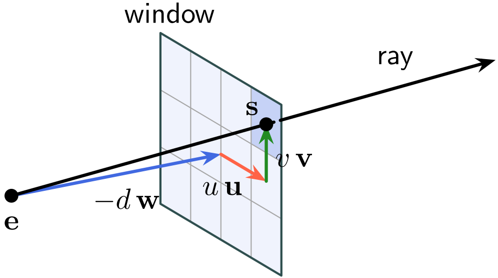


$$\mathbf{s} = \mathbf{e} - d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$$

three legs from $\mathbf{e}$

- Forward d, along $-\mathbf{w}$
- Across u, then up v

Walk from e to s in three legs



$$\mathbf{s} = \mathbf{e} - d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$$

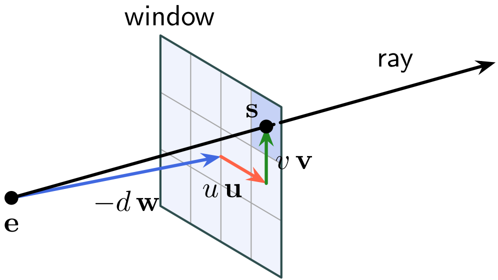
three legs from e

direction $= \mathbf{s} - \mathbf{e}$

the ray through s

- Forward d, along $-\mathbf{w}$
- Across u, then up v
- Straight there: the ray

Walk from e to s in three legs



$$\mathbf{s} = \mathbf{e} - d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$$

three legs from e

$$\text{direction} = \mathbf{s} - \mathbf{e}$$

the ray through s

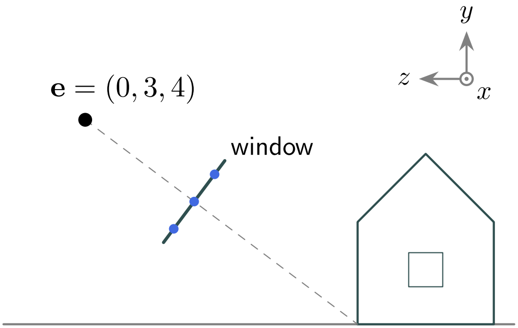
$$= -d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$$

e cancels

- Forward d, along $-\mathbf{w}$
- Across u, then up v
- Straight there: the ray

Worked: the ray through pixel $(3, 2)$

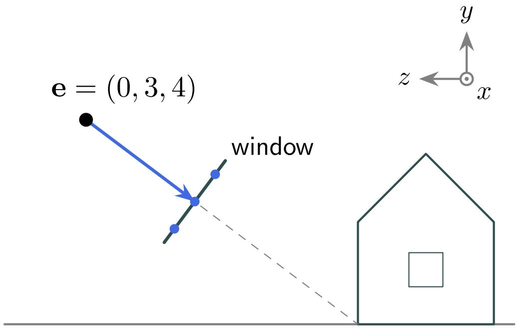
$$u = 0.75, \quad v = 0.5, \quad d = 2$$



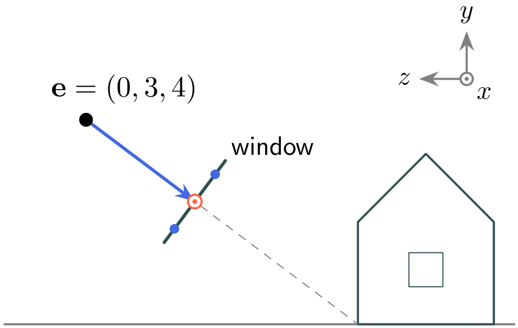
Worked: the ray through pixel $(3, 2)$

$$u = 0.75, \quad v = 0.5, \quad d = 2$$

$$-d\,\mathbf{w} = -2\,(0,\ 0.6,\ 0.8) = (0,\ -1.2,\ -1.6)$$



Worked: the ray through pixel $(3, 2)$

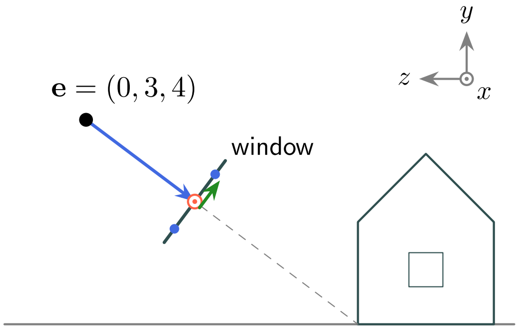


$$u = 0.75, \quad v = 0.5, \quad d = 2$$

$$-d\,\mathbf{w} = -2\,(0,\ 0.6,\ 0.8) = (0,\ -1.2,\ -1.6)$$

$$u\,\mathbf{u} = 0.75\,(1,\ 0,\ 0) = (0.75,\ 0,\ 0)$$

Worked: the ray through pixel $(3, 2)$



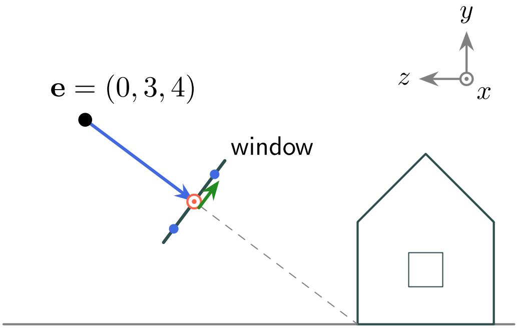
$$u = 0.75, \quad v = 0.5, \quad d = 2$$

$$-d\,\mathbf{w} = -2\,(0,\ 0.6,\ 0.8) = (0,\ -1.2,\ -1.6)$$

$$u\,\mathbf{u} = 0.75\,(1,\ 0,\ 0) = (0.75,\ 0,\ 0)$$

$$v\,\mathbf{v} = 0.5\,(0,\ 0.8,\ -0.6) = (0,\ 0.4,\ -0.3)$$

Worked: the ray through pixel $(3, 2)$



$$u = 0.75, \quad v = 0.5, \quad d = 2$$

$$-d\,\mathbf{w} = -2\,(0,\ 0.6,\ 0.8) = (0,\ -1.2,\ -1.6)$$

$$u\,\mathbf{u} = 0.75\,(1,\ 0,\ 0) = (0.75,\ 0,\ 0)$$

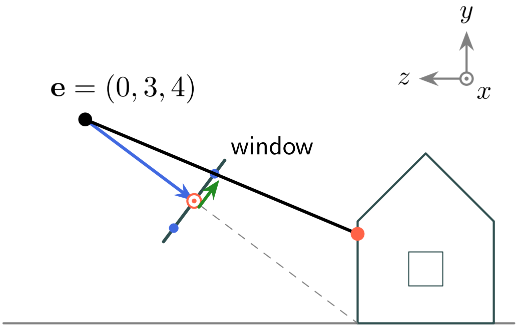
$$v\,\mathbf{v} = 0.5\,(0,\ 0.8,\ -0.6) = (0,\ 0.4,\ -0.3)$$

sum: $(0{+}0.75{+}0,\quad -1.2{+}0{+}0.4,\quad -1.6{+}0{-}0.3)$

direction $= (0.75,\ -0.8,\ -1.9)$

- Three scaled arrows, added up

Worked: the ray through pixel $(3, 2)$



$$u = 0.75, \quad v = 0.5, \quad d = 2$$

$$-d\,\mathbf{w} = -2\,(0,\ 0.6,\ 0.8) = (0,\ -1.2,\ -1.6)$$

$$u\,\mathbf{u} = 0.75\,(1,\ 0,\ 0) = (0.75,\ 0,\ 0)$$

$$v\,\mathbf{v} = 0.5\,(0,\ 0.8,\ -0.6) = (0,\ 0.4,\ -0.3)$$

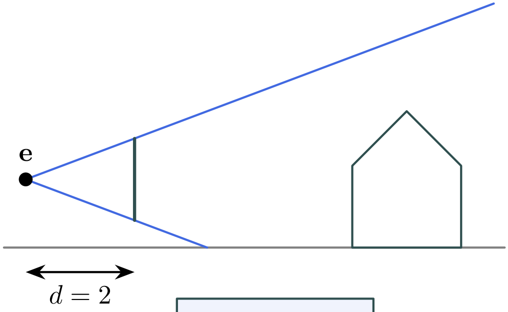
sum: $(0{+}0.75{+}0,\quad -1.2{+}0{+}0.4,\quad -1.6{+}0{-}0.3)$

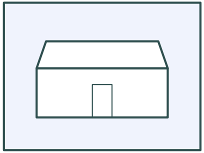
direction $= (0.75,\ -0.8,\ -1.9)$

$$\mathbf{p}(t) = (0, 3, 4) + t\,(0.75,\ -0.8,\ -1.9)$$

- Three scaled arrows, added up
- World numbers, ready to trace

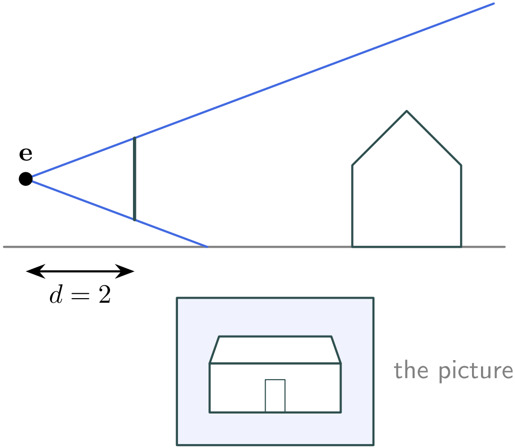
A longer d zooms in

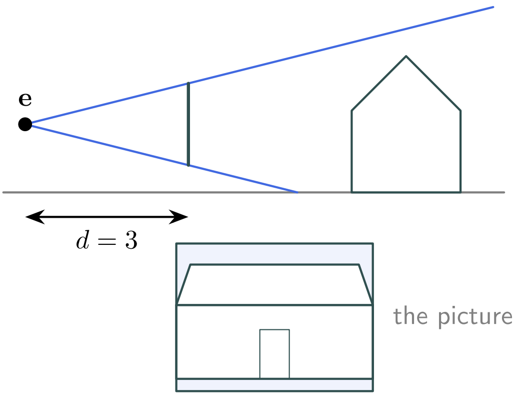




the picture

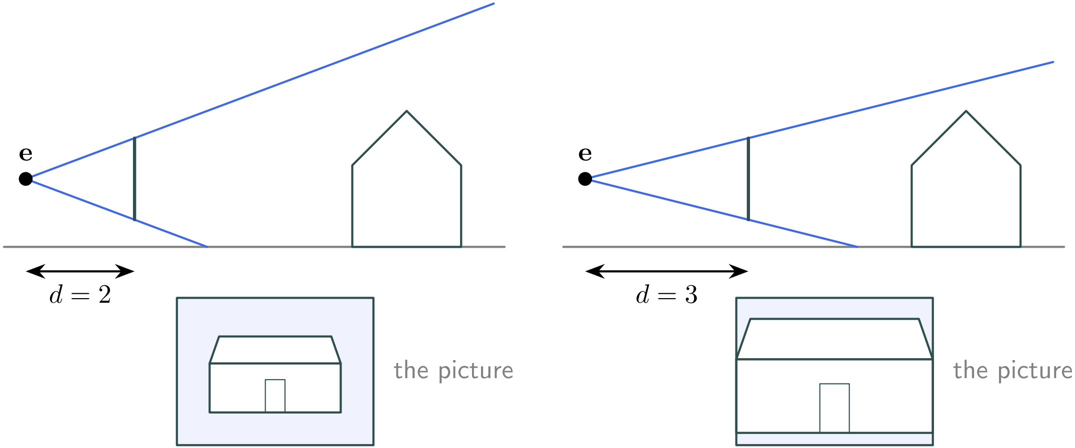
A longer d zooms in





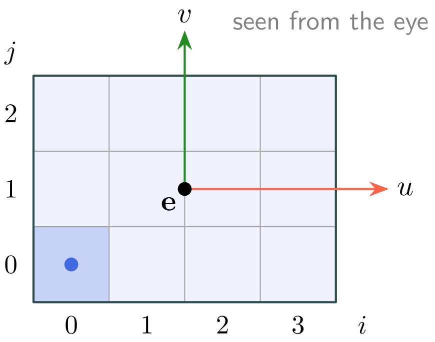
- Same window, farther out
- Narrow fan, big house

A longer d zooms in



- Same window, farther out
- Narrow fan, big house
- Like a longer focal length

Quick check 2: the ray through pixel $(0, 0)$



$$u = l + (r - l)\,\frac{i + 0.5}{n_x}$$

$$v = b + (t - b)\,\frac{j + 0.5}{n_y}$$

$$\text{direction} = -d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$$

$$l = -1, \quad r = 1, \quad b = -0.75, \quad t = 0.75$$

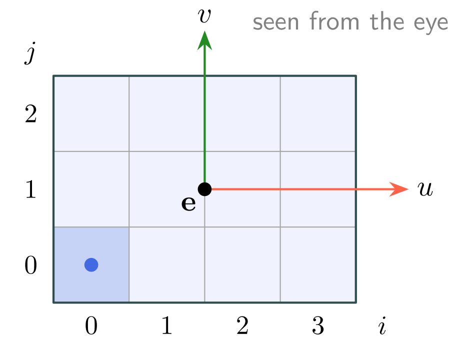
$$n_x = 4, \quad n_y = 3, \quad d = 2$$

$$\mathbf{u} = (1,\ 0,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$$

$$\mathbf{v} = (0,\ 0.8,\ -0.6)$$

- Find (u, v) for pixel $(0, 0)$
- Then its ray's direction

Quick check 2: the ray through pixel $(0, 0)$



$$u = l + (r - l)\,\frac{i + 0.5}{n_x}$$

$$v = b + (t - b)\,\frac{j + 0.5}{n_y}$$

direction $= -d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$

$l = -1, \quad r = 1, \quad b = -0.75, \quad t = 0.75$

$n_x = 4, \quad n_y = 3, \quad d = 2$

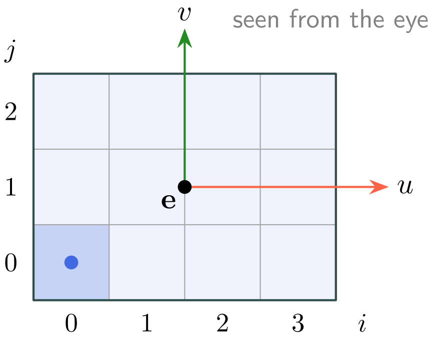
$\mathbf{u} = (1,\ 0,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$

$\mathbf{v} = (0,\ 0.8,\ -0.6)$

- Find (u, v) for pixel $(0, 0)$
- Then its ray's direction

$(u, v) = (-0.75,\ -0.5)$

Quick check 2: the ray through pixel $(0, 0)$



$$u = l + (r - l)\,\frac{i + 0.5}{n_x}$$

$$v = b + (t - b)\,\frac{j + 0.5}{n_y}$$

direction $= -d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$

$l = -1, \quad r = 1, \quad b = -0.75, \quad t = 0.75$

$n_x = 4, \quad n_y = 3, \quad d = 2$

$\mathbf{u} = (1,\ 0,\ 0), \quad \mathbf{w} = (0,\ 0.6,\ 0.8)$

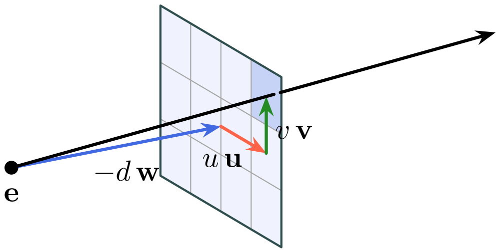
$\mathbf{v} = (0,\ 0.8,\ -0.6)$

- Find (u, v) for pixel $(0, 0)$
- Then its ray's direction

$(u, v) = (-0.75,\ -0.5)$

direction $= (-0.75,\ -1.6,\ -1.3)$

Recap: the ray through a pixel

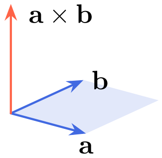


origin $= \mathbf{e}$

direction $= -d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$

Summary

1 Perpendicular to both, long as the area



x: $y_a z_b - z_a y_b$

y: $z_a x_b - x_a z_b$

z: $x_a y_b - y_a x_b$

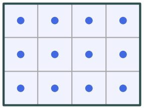
2 Two arrows in, three axes out

$$\mathbf{w} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

$$\mathbf{u} = \frac{\mathbf{b}\times\mathbf{w}}{\|\mathbf{b}\times\mathbf{w}\|}$$

$$\mathbf{v} = \mathbf{w}\times\mathbf{u}$$

3 Pixel centres: half a cell in

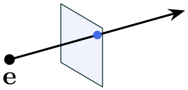


$$u = l + (r-l)\,\frac{i+0.5}{n_x}$$

$$v = b + (t-b)\,\frac{j+0.5}{n_y}$$

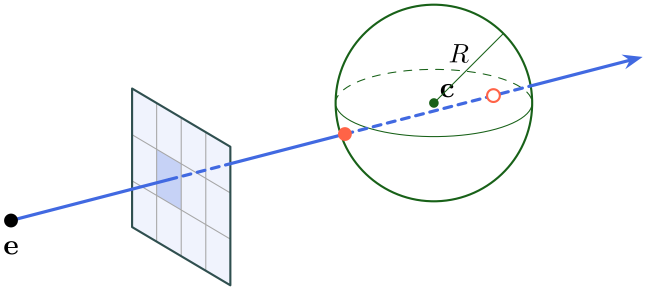
4 From the eye, through the pixel

origin $= \mathbf{e}$



direction $= -d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$

Next lecture: where a ray meets a sphere



- Same ray, a round target
- Where does it go in?
- A quadratic in t