Lecture 12: Exact Hits and Soft Shadows

Where a Ray Meets a Wall, and Lights with Size

CS 453 — Computer Graphics

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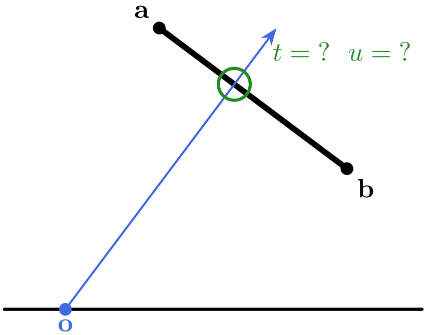
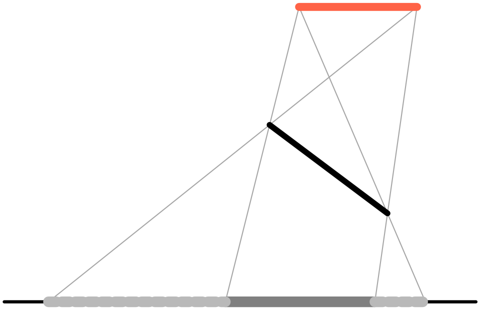
Forman Christian University

Today: exact hits, then soft shadows

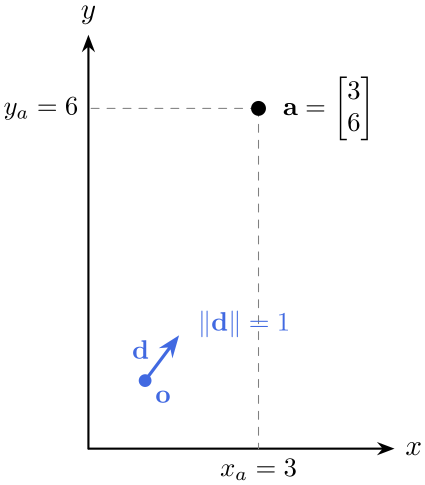


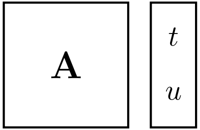
Exactly where does a ray hit?



Lights with size cast soft edges

Conventions: bold points, columns, y up



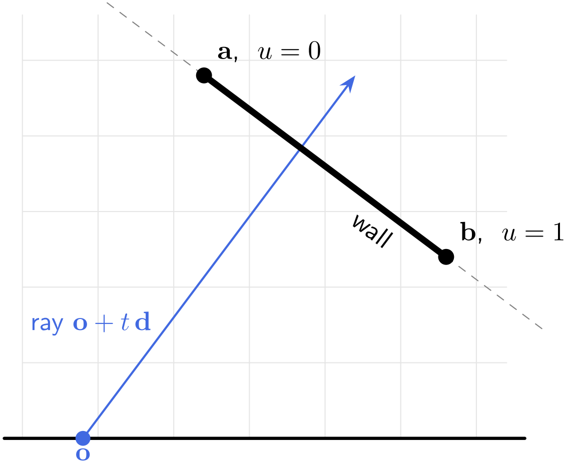


- Bold: points and vectors
- Columns, with y pointing up
- Matrix on the left of a column

Solving for t and u

Two equations, two unknowns, one rule

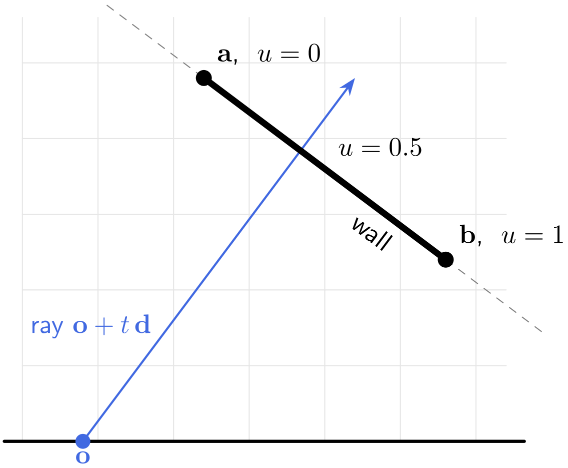
A wall: a segment from $\mathbf{a}$ to $\mathbf{b}$



$$\mathbf{a} + u\,(\mathbf{b} - \mathbf{a}), \quad 0 \leq u \leq 1$$

- Goal: where the ray hits it
- Ray owns t; wall gets u

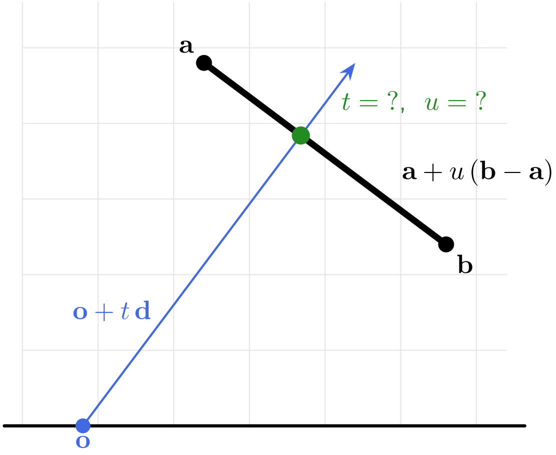
A wall: a segment from $\mathbf{a}$ to $\mathbf{b}$



$$\mathbf{a} + u\,(\mathbf{b} - \mathbf{a}), \quad 0 \le u \le 1$$

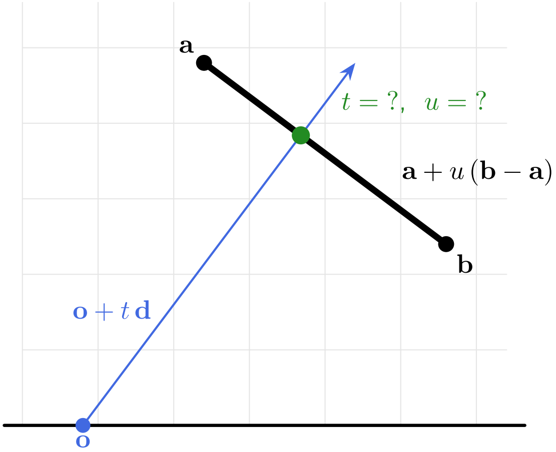
- Goal: where the ray hits it
- Ray owns t; wall gets u
- Only $0 \le u \le 1$ is wall

Where they cross, both formulas give one point



- One point, two addresses

Where they cross, both formulas give one point



- One point, two addresses
- Unknowns: t on the ray, u on the wall

$$\mathbf{o} + t\,\mathbf{d} = \mathbf{a} + u\,(\mathbf{b} - \mathbf{a})$$

The crossing equation is a 2×2 linear system

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$$t\,\mathbf{d} + u\,(\mathbf{a} - \mathbf{b}) = \mathbf{a} - \mathbf{o}$$

- Unknowns left, knowns right

The crossing equation is a 2×2 linear system

$$\mathbf{o} + t\,\mathbf{d} = \mathbf{a} + u\,(\mathbf{b} - \mathbf{a})$$

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$$\overbrace{\begin{bmatrix} \underbrace{\boxed{\begin{matrix} x_d \\ y_d \end{matrix}}}_{\mathbf{d}} & \underbrace{\boxed{\begin{matrix} x_a - x_b \\ y_a - y_b \end{matrix}}}_{\mathbf{a}-\mathbf{b}} \end{bmatrix}}^{\mathbf{A}} \begin{bmatrix} t \\ u \end{bmatrix} = \begin{bmatrix} \underbrace{\boxed{\begin{matrix} x_a - x_o \\ y_a - y_o \end{matrix}}}_{\mathbf{a}-\mathbf{o}} \end{bmatrix}$$

- Unknowns left, knowns right
- t, u mix the columns (lecture 7)

The crossing equation is a 2×2 linear system

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$$t\,\mathbf{d} + u\,(\mathbf{a} - \mathbf{b}) = \mathbf{a} - \mathbf{o}$$

$$\begin{array}{cc} & \mathbf{A} \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} \boxed{x_d} & \boxed{x_a - x_b} \\ \boxed{y_d} & \boxed{y_a - y_b} \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix} = \begin{bmatrix} \boxed{x_a - x_o} \\ \boxed{y_a - y_o} \end{bmatrix} \\ & \begin{array}{ccc} \mathbf{d} & \mathbf{a} - \mathbf{b} & \mathbf{a} - \mathbf{o} \end{array} \end{array}$$

- Unknowns left, knowns right
- t, u mix the columns (lecture 7)
- Rows: the x and y equations

The crossing equation is a 2×2 linear system

$$\mathbf{o} + t\,\mathbf{d} = \mathbf{a} + u\,(\mathbf{b} - \mathbf{a})$$

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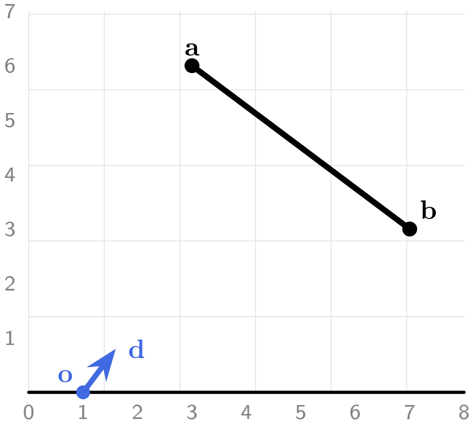
$$\begin{array}{c c c} & \mathbf{A} & \\ \begin{array}{c} x \\ y \end{array} & \begin{bmatrix} \boxed{\begin{matrix} x_d \\ y_d \end{matrix}} & \boxed{\begin{matrix} x_a - x_b \\ y_a - y_b \end{matrix}} \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} x_a - x_o \\ y_a - y_o \end{matrix}} \end{bmatrix} \\ & \quad \mathbf{d} \qquad\quad \mathbf{a} - \mathbf{b} \qquad\qquad\qquad\quad \mathbf{a} - \mathbf{o} & \end{array}$$

$$\boxed{\mathbf{A}\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] = \mathbf{a} - \mathbf{o}}$$

- Unknowns left, knowns right
- t, u mix the columns (lecture 7)
- Rows: the x and y equations
- Next: Cramer's rule finds t, u

Worked: our scene's numbers fill $\mathbf{A}$



$$\mathbf{o} = (1,\ 0), \quad \mathbf{d} = (0.6,\ 0.8)$$
$$\mathbf{a} = (3,\ 6), \quad \mathbf{b} = (7,\ 3)$$

- Read the points off the grid

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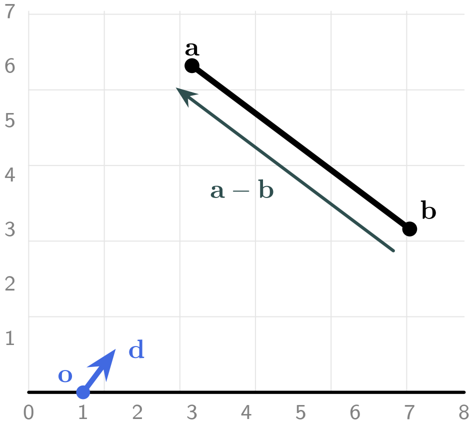

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$$\mathbf{a} - \mathbf{b} = (3 - 7,\ 6 - 3) = (-4,\ 3)$$

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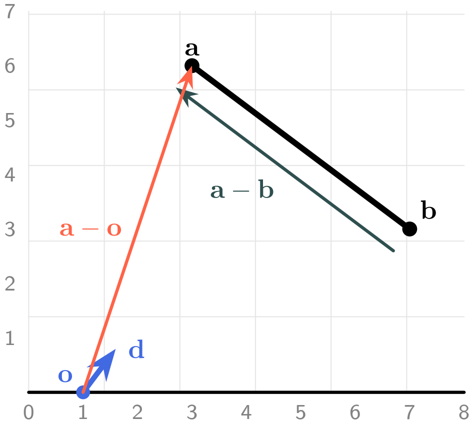

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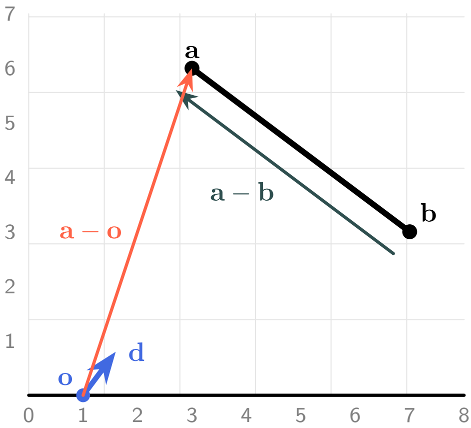

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$$\mathbf{a} - \mathbf{b} = (3-7,\ 6-3) = (-4,\ 3)$$
$$\mathbf{a} - \mathbf{o} = (3-1,\ 6-0) = (2,\ 6)$$

$$\begin{bmatrix} 0.6 & -4 \\ 0.8 & 3 \end{bmatrix} \begin{bmatrix} t \\ u \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

- Read the points off the grid
- Now solve this for t and u

A determinant turns four numbers into one

$$\begin{vmatrix} 0.6 & -4 \\ 0.8 & 3 \end{vmatrix}$$

$$|\mathbf{A}| = x_d\,(y_a - y_b) - (x_a - x_b)\,y_d$$

A determinant turns four numbers into one

$$\begin{vmatrix} 0.6 & -4 \\ 0.8 & 3 \end{vmatrix}$$

down-right

$$|\mathbf{A}| = x_d\,(y_a - y_b) - (x_a - x_b)\,y_d$$

down-right: $(0.6)(3) = 1.8$

- Multiply down the main diagonal

A determinant turns four numbers into one

$$\begin{vmatrix} 0.6 & -4 \\ 0.8 & 3 \end{vmatrix}$$

up-right

down-right

$$|\mathbf{A}| = x_d\,(y_a - y_b) - (x_a - x_b)\,y_d$$

down-right: $(0.6)(3) = 1.8$

up-right: $(-4)(0.8) = -3.2$

- Multiply down the main diagonal
- Subtract the other diagonal's product

A determinant turns four numbers into one

$$\begin{vmatrix} 0.6 & -4 \\ 0.8 & 3 \end{vmatrix}$$

up-right

down-right

$$|\mathbf{A}| = x_d\,(y_a - y_b) - (x_a - x_b)\,y_d$$

down-right: $(0.6)(3) = 1.8$

up-right: $(-4)(0.8) = -3.2$

$$|\mathbf{A}| = 1.8 - (-3.2) = 5$$

- Multiply down the main diagonal
- Subtract the other diagonal's product

Cramer's rule: each unknown is a determinant ratio

Cramer's rule

For equations written as $\mathbf{A}\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] =$ right side: each unknown is the determinant of $\mathbf{A}$ with that unknown's column replaced by the right side, divided by $|\mathbf{A}|$.

$$\mathbf{A} \quad \boxed{t\text{'s column}}\ \boxed{u\text{'s column}} \left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] = \boxed{\text{right side}}$$

known: $\mathbf{A}$ and the right side; unknown: t, u

$|\cdot|$ is a determinant (last slide): one number. Needs $|\mathbf{A}| \neq 0$. The same rule solves 3 unknowns. (FoCG §6.3.2)

Cramer's rule: each unknown is a determinant ratio

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known: $\mathbf{A}$ and the right side; unknown: t, u

$\mathbf{A}_t$ $\boxed{\text{right side}}$ $\boxed{u\text{'s column}}$ $t = |\mathbf{A}_t| \,/\, |\mathbf{A}|$

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Cramer's rule: each unknown is a determinant ratio

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$\mathbf{A}$ [t's column] [u's column] $\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] =$ [right side]

known: $\mathbf{A}$ and the right side; unknown: t, u

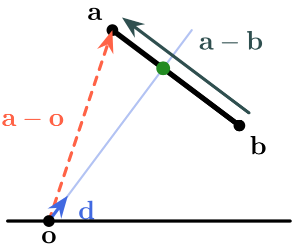
$\mathbf{A}_t$ [right side] [u's column] $t = |\mathbf{A}_t| \,/\, |\mathbf{A}|$

$\mathbf{A}_u$ [t's column] [right side] $u = |\mathbf{A}_u| \,/\, |\mathbf{A}|$

$|\cdot|$ is a determinant (last slide): one number. Needs $|\mathbf{A}| \neq 0$. The same rule solves 3 unknowns. (FoCG §6.3.2)

Cramer's rule on our ray and wall

Where does the ray hit the wall?



$$t\,\mathbf{d} + u\,(\mathbf{a}-\mathbf{b}) = \mathbf{a}-\mathbf{o}$$

the crossing equation: our $\mathbf{A}\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right]$ = right side

$$\mathbf{A}\ \begin{array}{cc} t\text{'s column} & u\text{'s column} \\ \boxed{\mathbf{d}} & \boxed{\mathbf{a}-\mathbf{b}} \end{array} \left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] = \begin{array}{c} \text{right side} \\ \boxed{\mathbf{a}-\mathbf{o}} \end{array}$$

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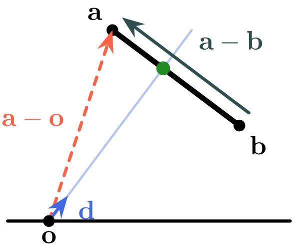

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the crossing equation: our $\mathbf{A}\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right]$ = right side

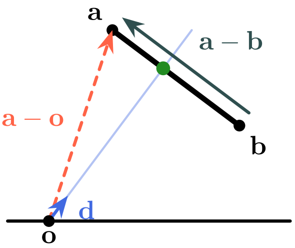
$$\begin{array}{ccccc} & t\text{'s column} & u\text{'s column} & & \text{right side} \\ \mathbf{A} & \boxed{\mathbf{d}} & \boxed{\mathbf{a}-\mathbf{b}} & \left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] = & \boxed{\mathbf{a}-\mathbf{o}} \end{array}$$

$$\mathbf{A}_t \quad \boxed{\mathbf{a}-\mathbf{o}} \quad \boxed{\mathbf{a}-\mathbf{b}} \qquad t = |\mathbf{A}_t|\,/\,|\mathbf{A}|$$

how far along the ray

Cramer's rule on our ray and wall

Where does the ray hit the wall?

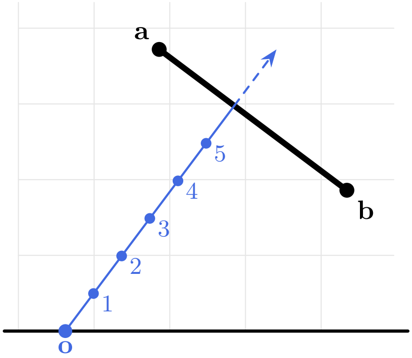


$$t\,\mathbf{d} + u\,(\mathbf{a} - \mathbf{b}) = \mathbf{a} - \mathbf{o}$$

the crossing equation: our $\mathbf{A}\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] =$ right side

	t's column	u's column		right side
$\mathbf{A}$	$\mathbf{d}$	$\mathbf{a} - \mathbf{b}$	$\left[\begin{smallmatrix} t \\ u \end{smallmatrix}\right] =$	$\mathbf{a} - \mathbf{o}$
$\mathbf{A}_t$	$\mathbf{a} - \mathbf{o}$	$\mathbf{a} - \mathbf{b}$	$t = \lvert\mathbf{A}_t\rvert \,/\, \lvert\mathbf{A}\rvert$ how far along the ray	
$\mathbf{A}_u$	$\mathbf{d}$	$\mathbf{a} - \mathbf{o}$	$u = \lvert\mathbf{A}_u\rvert \,/\, \lvert\mathbf{A}\rvert$ where along the wall	

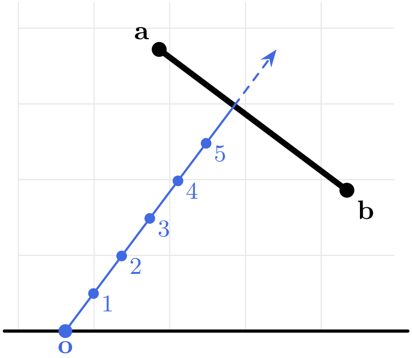
Worked: $t = 6$, six units along the ray



$$\mathbf{A}\quad \overset{\mathbf{d}}{\boxed{\begin{matrix}0.6\\0.8\end{matrix}}}\ \overset{\mathbf{a}-\mathbf{b}}{\boxed{\begin{matrix}-4\\3\end{matrix}}}\quad \begin{bmatrix}t\\u\end{bmatrix} = \overset{\mathbf{a}-\mathbf{o}}{\boxed{\begin{matrix}2\\6\end{matrix}}}$$

$$|\mathbf{A}| = 5$$

Worked: $t = 6$, six units along the ray



$$\mathbf{A} \quad \begin{array}{cc} \mathbf{d} & \mathbf{a}-\mathbf{b} \\ \boxed{\begin{matrix} 0.6 \\ 0.8 \end{matrix}} & \boxed{\begin{matrix} -4 \\ 3 \end{matrix}} \end{array} \quad \begin{bmatrix} t \\ u \end{bmatrix} = \begin{array}{c} \mathbf{a}-\mathbf{o} \\ \boxed{\begin{matrix} 2 \\ 6 \end{matrix}} \end{array}$$

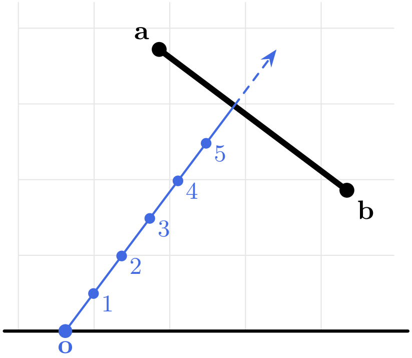
$$|\mathbf{A}| = 5$$

$$\mathbf{A}_t \quad \boxed{\begin{matrix} 2 \\ 6 \end{matrix}} \quad \boxed{\begin{matrix} -4 \\ 3 \end{matrix}}$$

$\mathbf{a} - \mathbf{o}$ replaces $\mathbf{d}$

$$|\mathbf{A}_t| = 2 \cdot 3 - (-4) \cdot 6 = 30$$

Worked: $t = 6$, six units along the ray



$$\mathbf{A} \quad \begin{array}{cc} \mathbf{d} & \mathbf{a}-\mathbf{b} \\ \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} & \begin{bmatrix} -4 \\ 3 \end{bmatrix} \end{array} \begin{bmatrix} t \\ u \end{bmatrix} = \begin{array}{c} \mathbf{a}-\mathbf{o} \\ \begin{bmatrix} 2 \\ 6 \end{bmatrix} \end{array}$$

$|\mathbf{A}| = 5$

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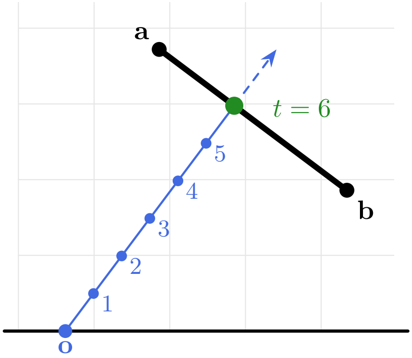
$\mathbf{a} - \mathbf{o}$ replaces $\mathbf{d}$

$|\mathbf{A}_t| = 2 \cdot 3 - (-4) \cdot 6 = 30$

$$t = \frac{|\mathbf{A}_t|}{|\mathbf{A}|} = \frac{30}{5} = 6$$

Worked: $t = 6$, six units along the ray



$$\mathbf{A} \quad \begin{array}{cc} \mathbf{d} & \mathbf{a}-\mathbf{b} \\ \boxed{\begin{matrix}0.6\\0.8\end{matrix}} & \boxed{\begin{matrix}-4\\3\end{matrix}} \end{array} \quad \begin{bmatrix} t \\ u \end{bmatrix} = \begin{array}{c} \mathbf{a}-\mathbf{o} \\ \boxed{\begin{matrix}2\\6\end{matrix}} \end{array}$$

$|\mathbf{A}| = 5$

$$\mathbf{A}_t \quad \boxed{\begin{matrix}2\\6\end{matrix}} \quad \boxed{\begin{matrix}-4\\3\end{matrix}}$$

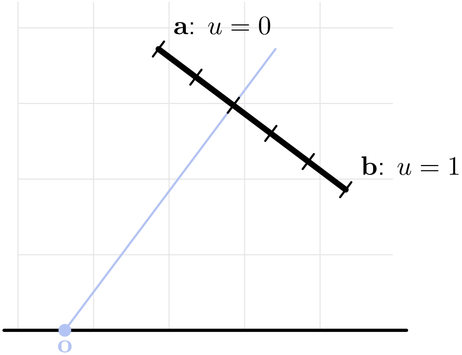
$\mathbf{a}-\mathbf{o}$ replaces $\mathbf{d}$

$|\mathbf{A}_t| = 2 \cdot 3 - (-4) \cdot 6 = 30$

$$t = \frac{|\mathbf{A}_t|}{|\mathbf{A}|} = \frac{30}{5} = 6$$

$\|\mathbf{d}\| = 1$: the hit is 6 units from $\mathbf{o}$

Worked: $u = 0.4$ of the way to $\mathbf{b}$



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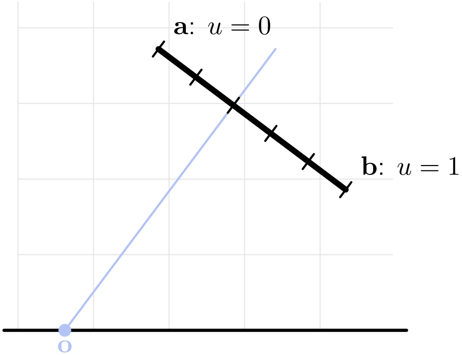
Worked: $u = 0.4$ of the way to $\mathbf{b}$



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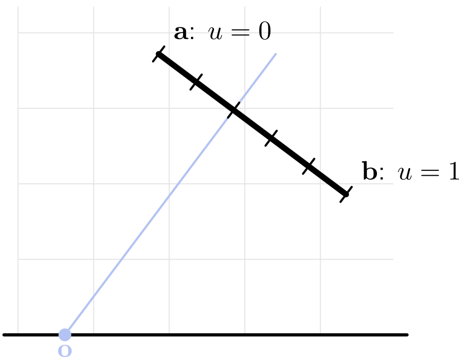
$|\mathbf{A}| = 5$

$$\mathbf{A}_u \quad \begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

$\mathbf{a} - \mathbf{o}$ replaces $\mathbf{a} - \mathbf{b}$

$|\mathbf{A}_u| = 0.6 \cdot 6 - 2 \cdot 0.8 = 2$

Worked: $u = 0.4$ of the way to $\mathbf{b}$



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$$|\mathbf{A}| = 5$$

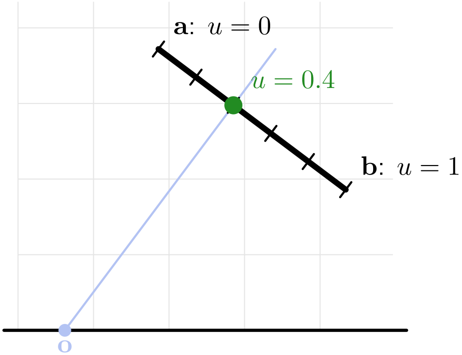
$$\mathbf{A}_u \quad \boxed{\begin{matrix} 0.6 \\ 0.8 \end{matrix}} \quad \boxed{\begin{matrix} 2 \\ 6 \end{matrix}}$$

$\mathbf{a}-\mathbf{o}$ replaces $\mathbf{a}-\mathbf{b}$

$$|\mathbf{A}_u| = 0.6 \cdot 6 - 2 \cdot 0.8 = 2$$

$$u = \frac{|\mathbf{A}_u|}{|\mathbf{A}|} = \frac{2}{5} = 0.4$$

Worked: $u = 0.4$ of the way to $\mathbf{b}$



$$\mathbf{A} \quad \overset{\mathbf{d}}{\boxed{\begin{matrix}0.6\\0.8\end{matrix}}} \; \overset{\mathbf{a}-\mathbf{b}}{\boxed{\begin{matrix}-4\\3\end{matrix}}} \quad \left[\begin{smallmatrix}t\\u\end{smallmatrix}\right] = \overset{\mathbf{a}-\mathbf{o}}{\boxed{\begin{matrix}2\\6\end{matrix}}}$$

$$|\mathbf{A}| = 5$$

$$\mathbf{A}_u \quad \boxed{\begin{matrix}0.6\\0.8\end{matrix}} \; \boxed{\begin{matrix}2\\6\end{matrix}}$$

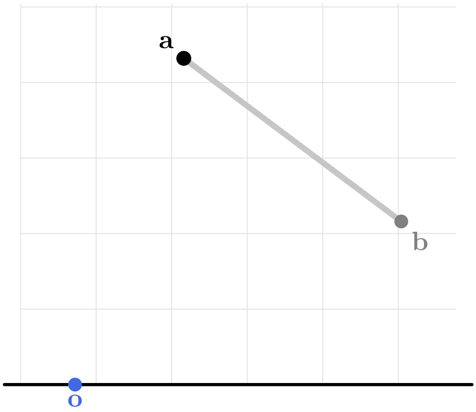
$\mathbf{a} - \mathbf{o}$ replaces $\mathbf{a} - \mathbf{b}$

$$|\mathbf{A}_u| = 0.6 \cdot 6 - 2 \cdot 0.8 = 2$$

$$u = \frac{|\mathbf{A}_u|}{|\mathbf{A}|} = \frac{2}{5} = 0.4$$

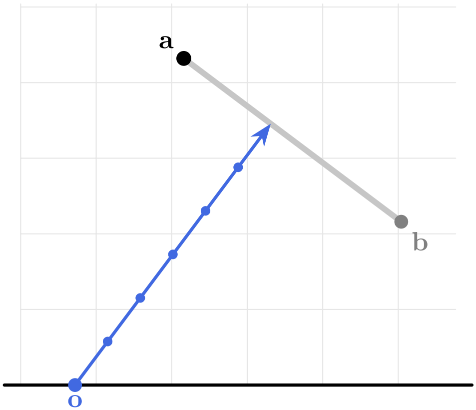
$0 \leq 0.4 \leq 1$: on the wall, 2 of its 5 units from $\mathbf{a}$

Check: both formulas land on $(4.6,\ 4.8)$



- Put t in the ray, u in the wall

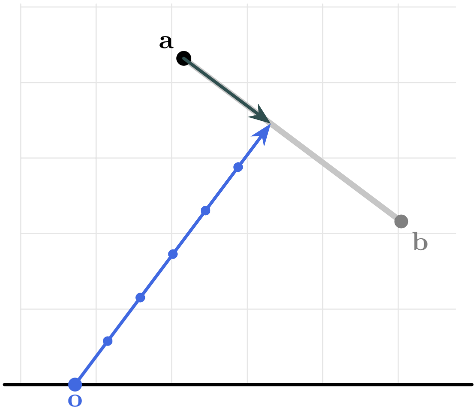
Check: both formulas land on $(4.6,\ 4.8)$



$$\begin{aligned}\mathbf{o} + 6\,\mathbf{d} &= (1,\ 0) + 6\,(0.6,\ 0.8) \\ &= (4.6,\ 4.8)\end{aligned}$$

- Put t in the ray, u in the wall
- From $\mathbf{o}$: six steps of $\mathbf{d}$

Check: both formulas land on $(4.6,\ 4.8)$



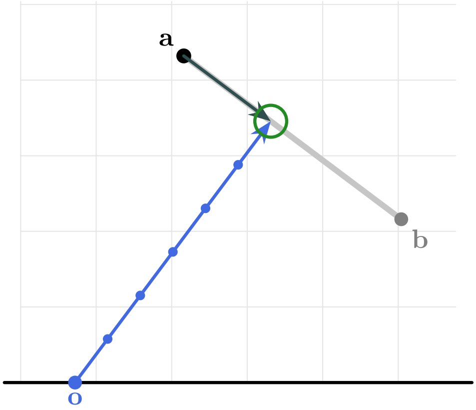
$$\begin{aligned}\mathbf{o} + 6\,\mathbf{d} &= (1,\ 0) + 6\,(0.6,\ 0.8)\\ &= (4.6,\ 4.8)\end{aligned}$$

$$\begin{aligned}\mathbf{a} + 0.4\,(\mathbf{b} - \mathbf{a}) &= (3,\ 6) + 0.4\,(4,\ -3)\\ &= (4.6,\ 4.8)\end{aligned}$$

using $\mathbf{b} - \mathbf{a} = (4,\ -3)$

- Put t in the ray, u in the wall
- From $\mathbf{o}$: six steps of $\mathbf{d}$
- From $\mathbf{a}$: 0.4 of the wall

Check: both formulas land on $(4.6,\ 4.8)$



$$\begin{aligned}\mathbf{o} + 6\,\mathbf{d} &= (1,\ 0) + 6\,(0.6,\ 0.8)\\ &= (4.6,\ 4.8)\end{aligned}$$

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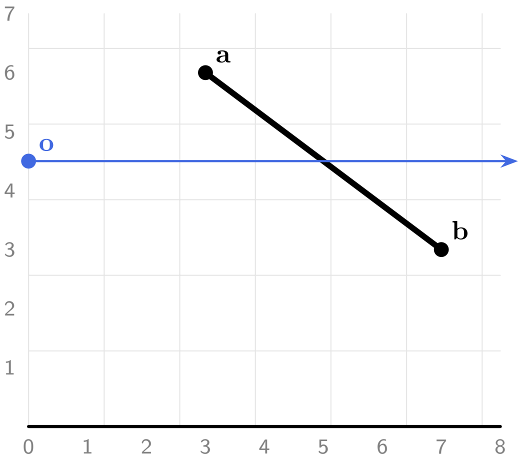
using $\mathbf{b} - \mathbf{a} = (4,\ -3)$

- Put t in the ray, u in the wall
- From $\mathbf{o}$: six steps of $\mathbf{d}$
- From $\mathbf{a}$: 0.4 of the wall
- Same point: t and u check out

Quick check 1: a level ray

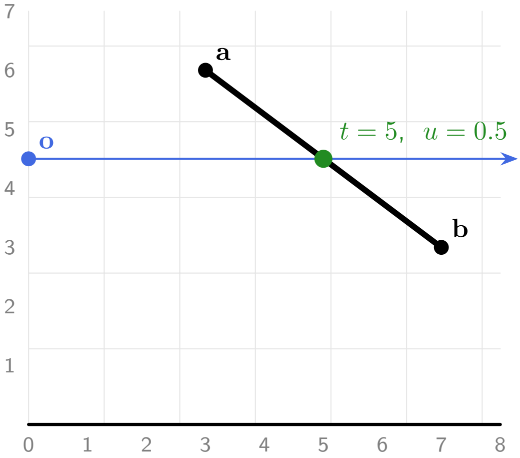


$$\mathbf{o} = (0,\ 4.5), \quad \mathbf{d} = (1,\ 0)$$

$$\mathbf{a} = (3,\ 6), \quad \mathbf{b} = (7,\ 3)$$

- Find $|\mathbf{A}|$, then t and u
- Check both on the picture

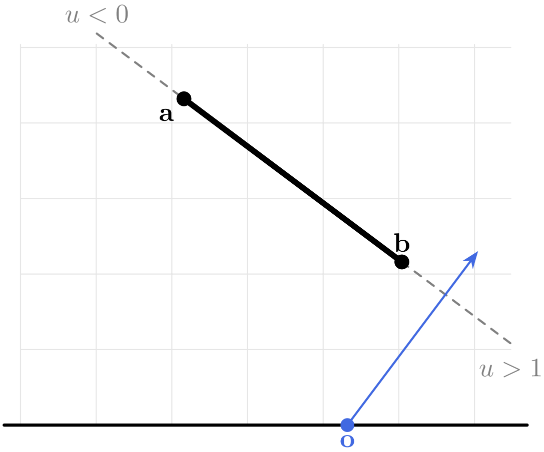
Quick check 1: a level ray



$$\mathbf{o} = (0,\ 4.5), \quad \mathbf{d} = (1,\ 0)$$
$$\mathbf{a} = (3,\ 6), \quad \mathbf{b} = (7,\ 3)$$

- Find $|\mathbf{A}|$, then t and u
- Check both on the picture
- $|\mathbf{A}| = 3, \ \ t = 15/3 = 5,$
  $u = 1.5/3 = 0.5$

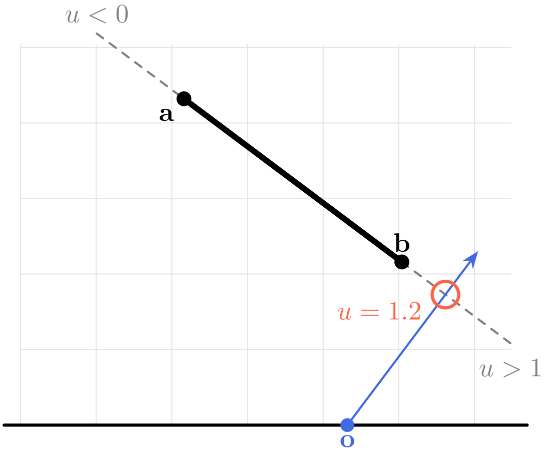
$u > 1$: the ray slips past the wall



$\mathbf{o} = (6,\ 0)$, same $\mathbf{d}$ and wall

$$\mathbf{a} - \mathbf{o} = (-3,\ 6), \quad |\mathbf{A}| = 5$$
$$t = 15/5 = 3$$
$$u = 6/5 = 1.2$$

$u > 1$: the ray slips past the wall

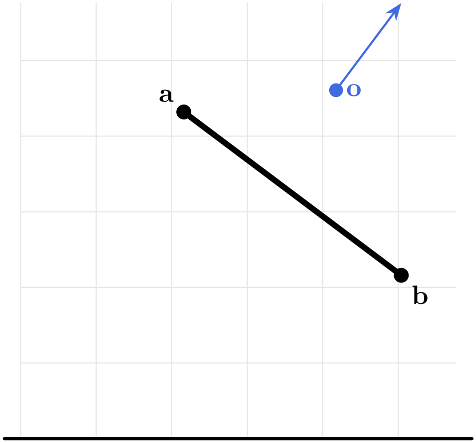


$\mathbf{o} = (6,\ 0)$, same $\mathbf{d}$ and wall

$$\mathbf{a} - \mathbf{o} = (-3,\ 6), \quad |\mathbf{A}| = 5$$
$$t = 15/5 = 3$$
$$u = 6/5 = 1.2$$

- Lines cross at $u = 1.2$: past $\mathbf{b}$
- Off the wall: a miss

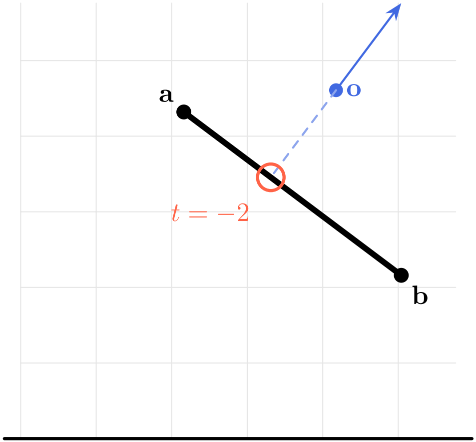
$t < 0$: the wall is behind the ray



$\mathbf{o} = (5.8,\ 6.4)$, same $\mathbf{d}$ and wall

$$\mathbf{a} - \mathbf{o} = (-2.8,\ -0.4), \quad |\mathbf{A}| = 5$$
$$t = -10/5 = -2$$
$$u = 2/5 = 0.4$$

$t < 0$: the wall is behind the ray

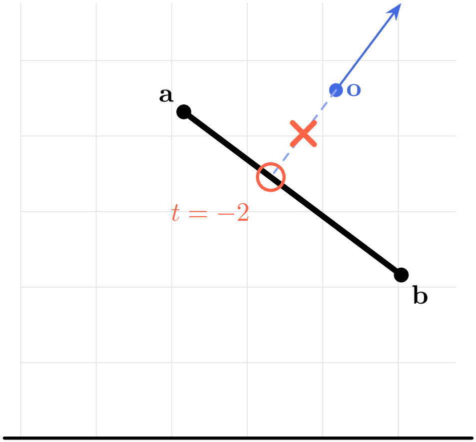


$\mathbf{o} = (5.8,\ 6.4)$, same $\mathbf{d}$ and wall

$$\mathbf{a} - \mathbf{o} = (-2.8,\ -0.4), \quad |\mathbf{A}| = 5$$
$$t = -10/5 = -2$$
$$u = 2/5 = 0.4$$

- $u = 0.4$: the wall is there
- $t = -2$: behind the start

$t < 0$: the wall is behind the ray

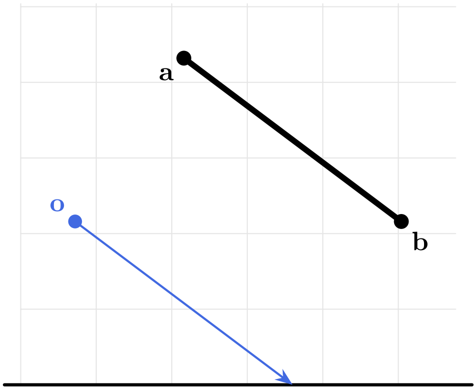


$$\mathbf{o} = (5.8,\ 6.4),\ \text{same } \mathbf{d} \text{ and wall}$$

$$\mathbf{a} - \mathbf{o} = (-2.8,\ -0.4), \quad |\mathbf{A}| = 5$$
$$t = -10/5 = -2$$
$$u = 2/5 = 0.4$$

- $u = 0.4$: the wall is there
- $t = -2$: behind the start
- A ray looks only forward: miss

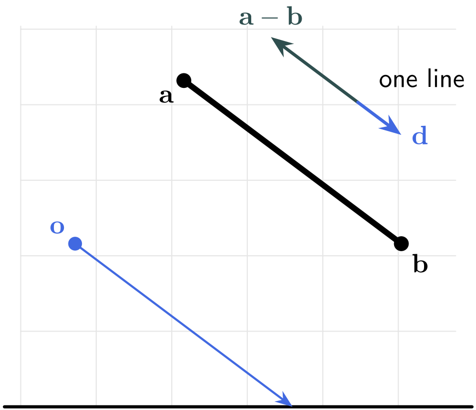
$|\mathbf{A}| = 0$: parallel, so no single answer



$$\mathbf{d} = (0.8,\ -0.6)$$
$$\mathbf{a} - \mathbf{b} = (-4,\ 3) = -5\,\mathbf{d}$$
$$|\mathbf{A}| = (0.8)(3) - (-4)(-0.6)$$
$$= 2.4 - 2.4 = 0$$

- $\mathbf{d}$ runs along the wall

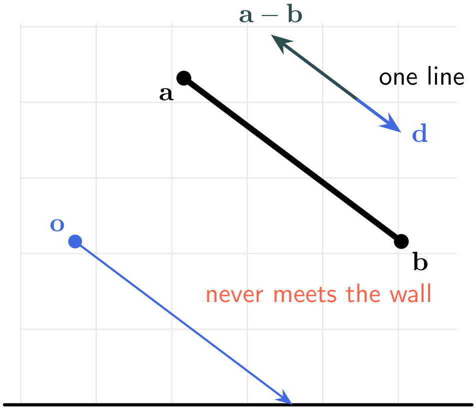
$|\mathbf{A}| = 0$: parallel, so no single answer



$$
\begin{aligned}
\mathbf{d} &= (0.8,\ -0.6) \\
\mathbf{a} - \mathbf{b} &= (-4,\ 3) = -5\,\mathbf{d} \\
|\mathbf{A}| &= (0.8)(3) - (-4)(-0.6) \\
&= 2.4 - 2.4 = 0
\end{aligned}
$$

- $\mathbf{d}$ runs along the wall
- Columns on one line: $|\mathbf{A}| = 0$

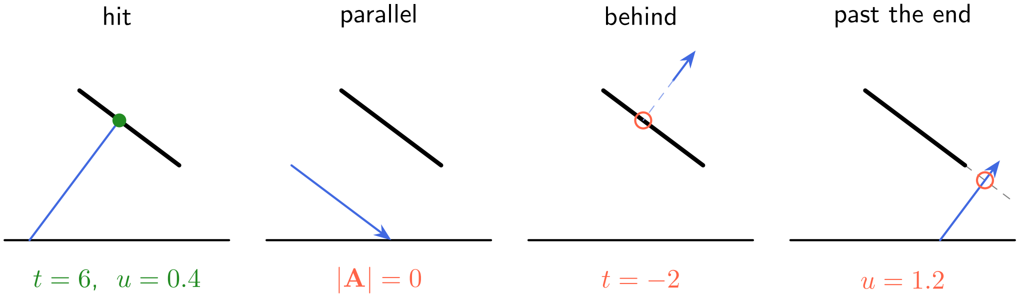
$|\mathbf{A}| = 0$: parallel, so no single answer



$$\mathbf{d} = (0.8,\ -0.6)$$
$$\mathbf{a} - \mathbf{b} = (-4,\ 3) = -5\,\mathbf{d}$$
$$|\mathbf{A}| = (0.8)(3) - (-4)(-0.6) = 2.4 - 2.4 = 0$$

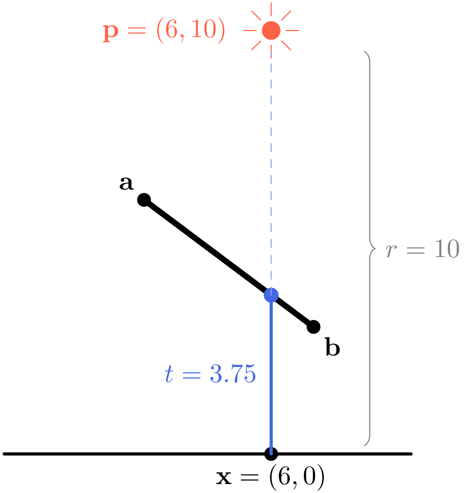
- $\mathbf{d}$ runs along the wall
- Columns on one line: $|\mathbf{A}| = 0$
- Cramer would divide by zero

A hit must pass all three checks



- $|\mathbf{A}| \neq 0$: not parallel
- $t \geq 0$: in front of $\mathbf{o}$
- $0 \leq u \leq 1$: on the wall

The shadow test is now exact



$$\mathbf{o} = \mathbf{x} = (6,\ 0), \quad r = 10$$
$$\mathbf{d} = \mathbf{l} = (0,\ 1)$$
$$\mathbf{a} - \mathbf{x} = (-3,\ 6), \quad |\mathbf{A}| = 4$$
$$t = 15/4 = 3.75$$
$$u = 3/4 = 0.75$$

- Shadow ray: $\mathbf{o} = \mathbf{x}$, $\mathbf{d} = \mathbf{l}$
- A hit in $[\epsilon, r]$ on the wall: shadow

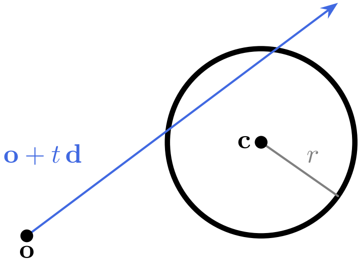
Recap: Cramer's rule for a ray and a wall

$$\begin{bmatrix}\mathbf{d} & \mathbf{a}-\mathbf{b}\end{bmatrix}\begin{bmatrix}t\\u\end{bmatrix} = \mathbf{a}-\mathbf{o}$$

$$t = \frac{\begin{vmatrix}\mathbf{a}-\mathbf{o} & \mathbf{a}-\mathbf{b}\end{vmatrix}}{\begin{vmatrix}\mathbf{d} & \mathbf{a}-\mathbf{b}\end{vmatrix}} \qquad u = \frac{\begin{vmatrix}\mathbf{d} & \mathbf{a}-\mathbf{o}\end{vmatrix}}{\begin{vmatrix}\mathbf{d} & \mathbf{a}-\mathbf{b}\end{vmatrix}}$$

hit: $|\mathbf{A}| \neq 0, \qquad t \geq 0, \qquad 0 \leq u \leq 1$

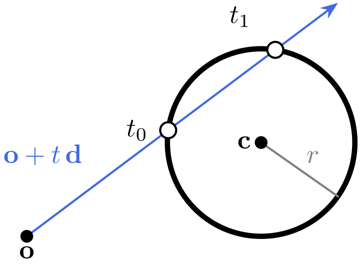
Ray meets circle: a quadratic in t



On the circle: r from the centre $\mathbf{c}$

$$\Big\| \underbrace{\mathbf{o} + t\,\mathbf{d}}_{\text{ray's point}} - \mathbf{c} \Big\| = r$$

Ray meets circle: a quadratic in t



On the circle: r from the centre $\mathbf{c}$

$$\Big\| \underbrace{\mathbf{o} + t\,\mathbf{d}}_{\text{ray's point}} - \mathbf{c} \Big\| = r$$

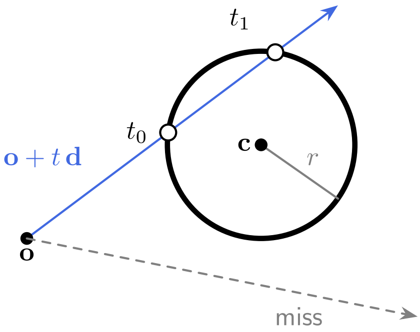
Square, expand, and use $\mathbf{d} \cdot \mathbf{d} = 1$:

$$t^2 + \beta\,t + \gamma = 0$$

$$\beta = 2\,\mathbf{d} \cdot (\mathbf{o} - \mathbf{c}), \qquad \gamma = \|\mathbf{o} - \mathbf{c}\|^2 - r^2$$

$$t_0,\ t_1 = \frac{-\beta \pm \sqrt{\beta^2 - 4\gamma}}{2}$$

Ray meets circle: a quadratic in t



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$$\Big\| \underbrace{\mathbf{o} + t\,\mathbf{d}}_{\text{ray's point}} - \mathbf{c} \Big\| = r$$

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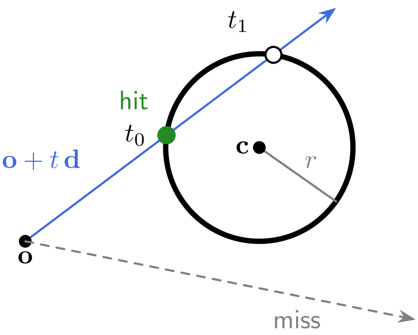
$\beta^2 - 4\gamma < 0$: no roots, a miss

Ray meets circle: a quadratic in t



On the circle: r from the centre $\mathbf{c}$

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$\beta^2 - 4\gamma < 0$: no roots, a miss

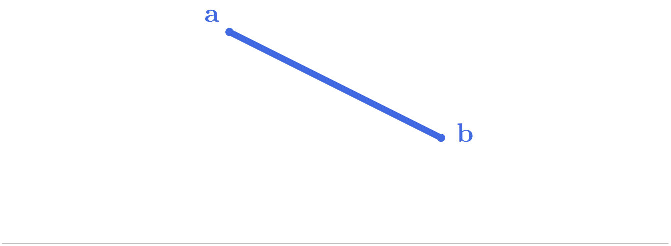
Hit: the smaller root with $t \geq 0$

Soft shadows

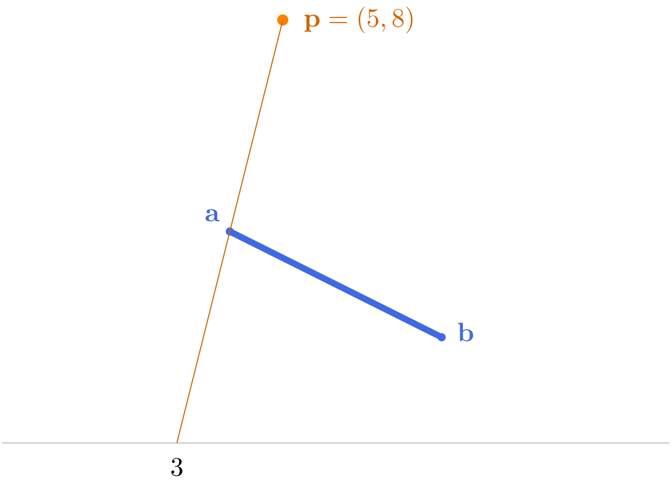
Cook, Porter & Carpenter, “Distributed Ray Tracing”, 1984

A point light casts a hard-edged shadow

- $\mathbf{p} = (5, 8)$

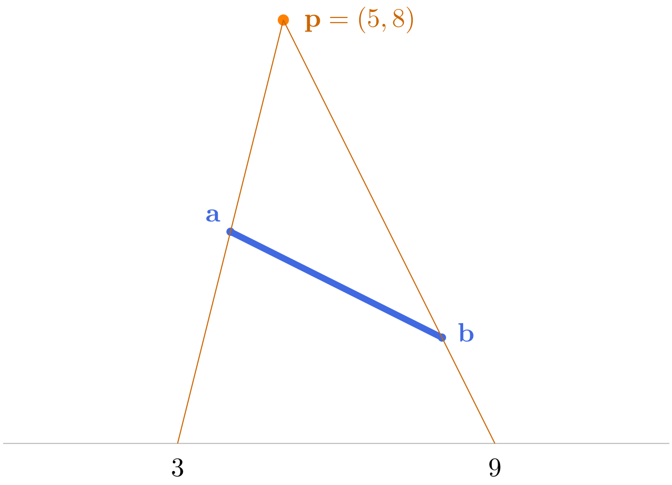


A point light casts a hard-edged shadow



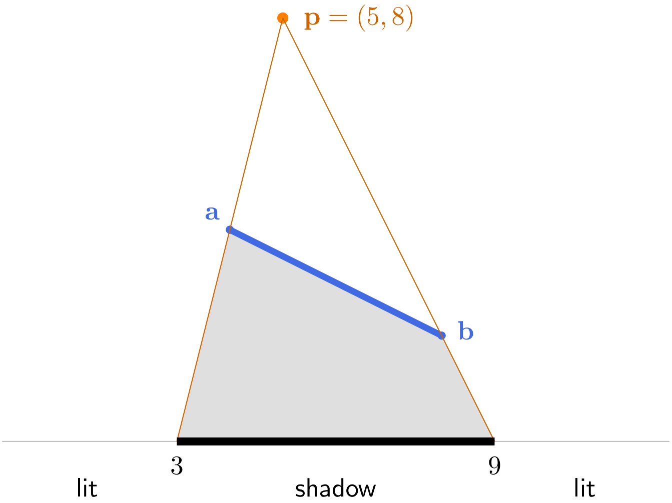
- The ray through $\mathbf{a}$ lands at 3

A point light casts a hard-edged shadow



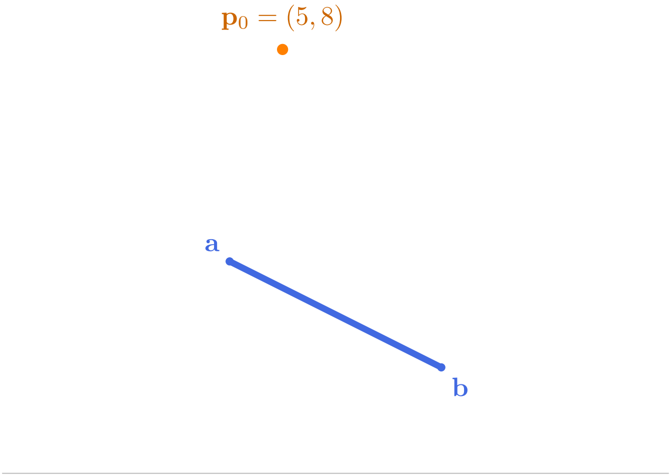
- The ray through $\mathbf{a}$ lands at 3
- The ray through $\mathbf{b}$ lands at 9

A point light casts a hard-edged shadow



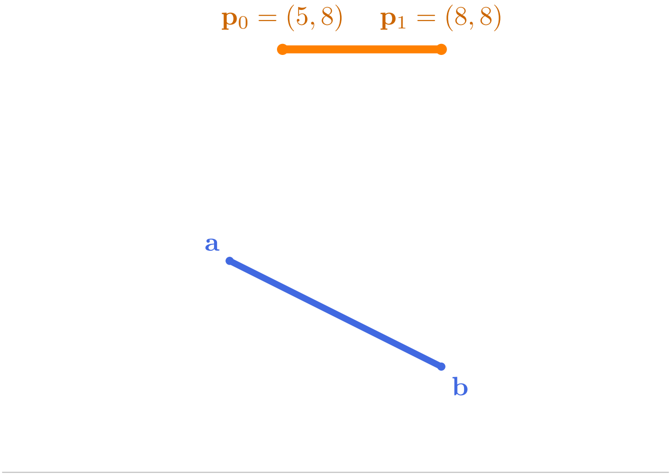
- The ray through $\mathbf{a}$ lands at 3
- The ray through $\mathbf{b}$ lands at 9
- Between: dark. Outside: lit. Nothing else

A long light gives the shadow soft edges



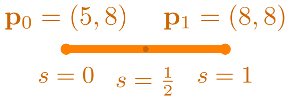
- One point: the old light

A long light gives the shadow soft edges

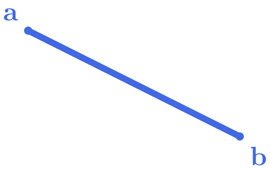


- One point: the old light
- A lamp is a segment too

A long light gives the shadow soft edges

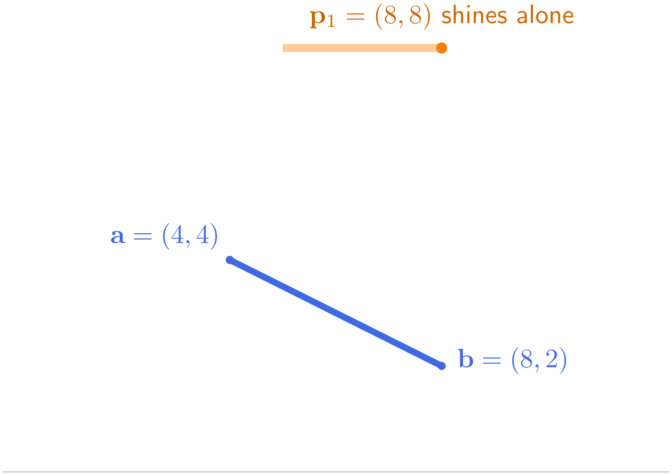


$$\mathbf{p}(s) = \mathbf{p}_0 + s\,(\mathbf{p}_1 - \mathbf{p}_0), \qquad 0 \le s \le 1$$



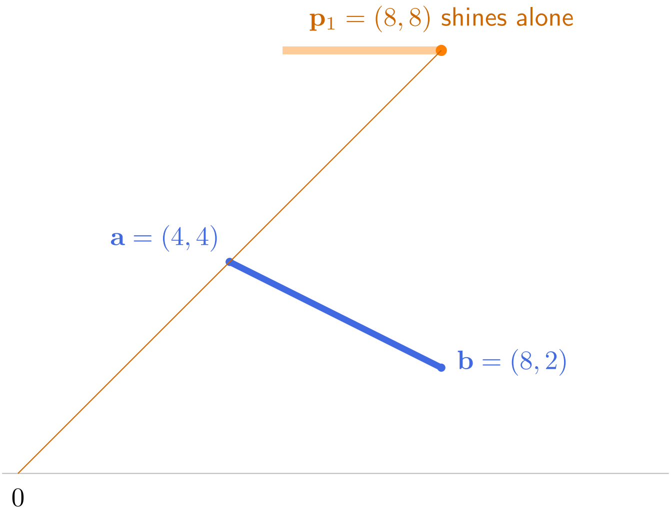
- One point: the old light
- A lamp is a segment too
- s picks a point on it: the wall's formula

Quick check 2: the right end's shadow



Where does $\mathbf{p}_1$'s shadow fall on the floor?

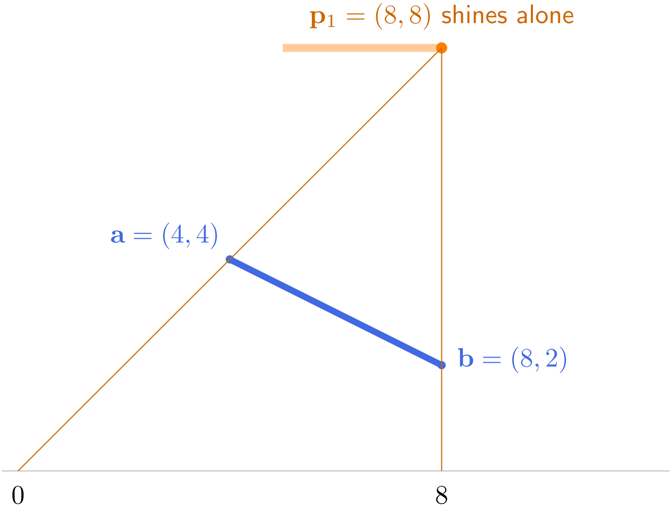
Quick check 2: the right end's shadow



Where does $\mathbf{p}_1$'s shadow fall on the floor?

Through $\mathbf{a}$: $(8,8) + s\,(-4,-4)$,
$y = 0$ at $s = 2$: $(0,0)$

Quick check 2: the right end's shadow

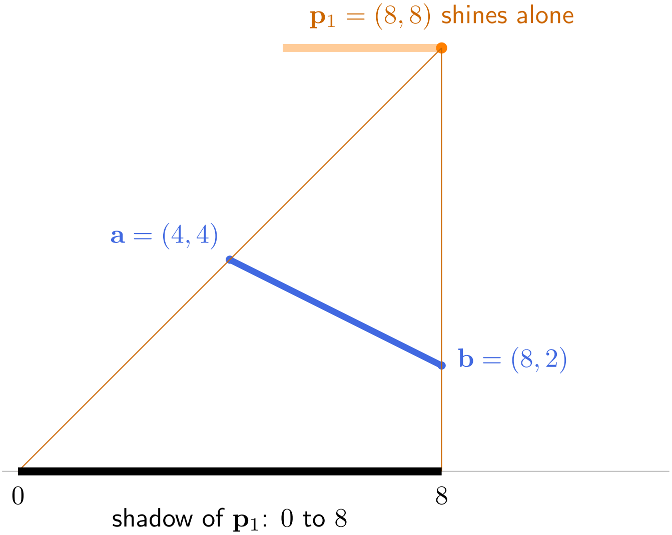


Where does $\mathbf{p}_1$'s shadow fall on the floor?

Through $\mathbf{a}$: $(8, 8) + s\,(-4, -4)$, $y = 0$ at $s = 2$: $(0, 0)$

Through $\mathbf{b}$: $(8, 8) + s\,(0, -6)$, $y = 0$ at $s = \frac{4}{3}$: $(8, 0)$

Quick check 2: the right end's shadow



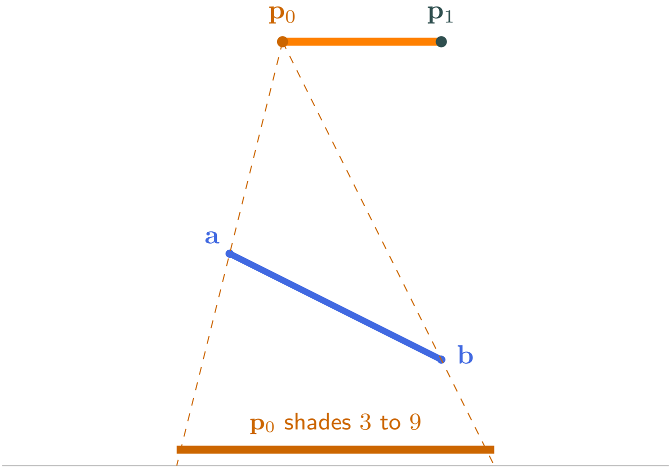
Where does $\mathbf{p}_1$'s shadow fall on the floor?

Through $\mathbf{a}$: $(8, 8) + s\,(-4, -4)$, $y = 0$ at $s = 2$: $(0, 0)$

Through $\mathbf{b}$: $(8, 8) + s\,(0, -6)$, $y = 0$ at $s = \frac{4}{3}$: $(8, 0)$

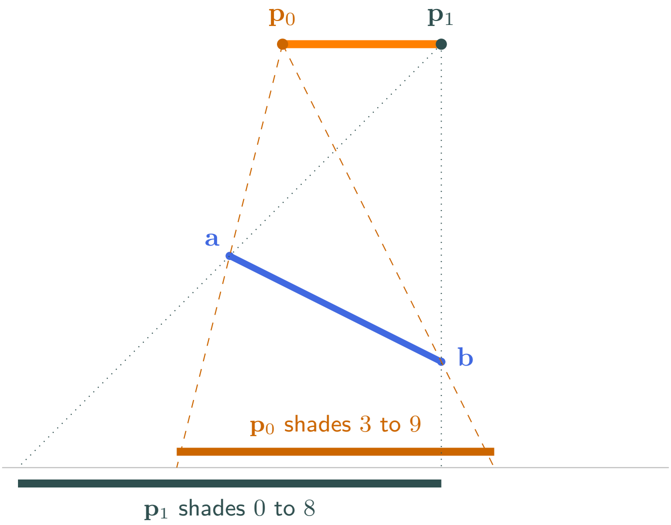
Dark from 0 to 8

Each end of the lamp shades differently



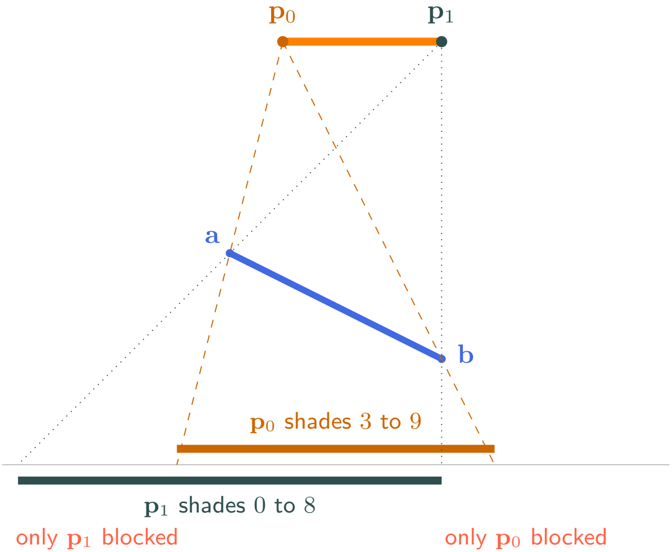
- Left end: dark from 3 to 9

Each end of the lamp shades differently



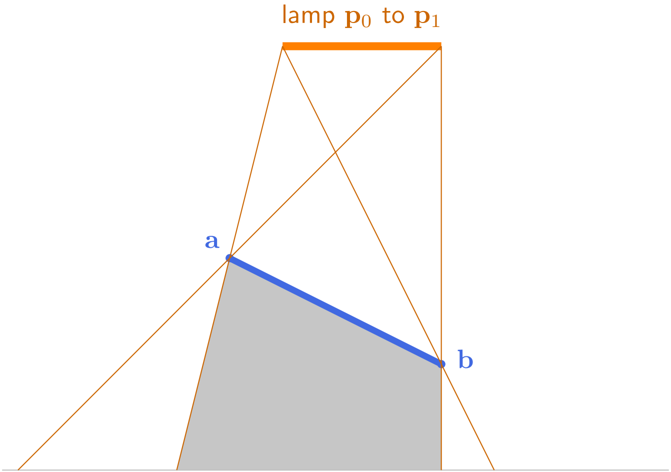
- Left end: dark from 3 to 9
- Right end: dark from 0 to 8

Each end of the lamp shades differently



- Left end: dark from 3 to 9
- Right end: dark from 0 to 8
- They disagree near both edges

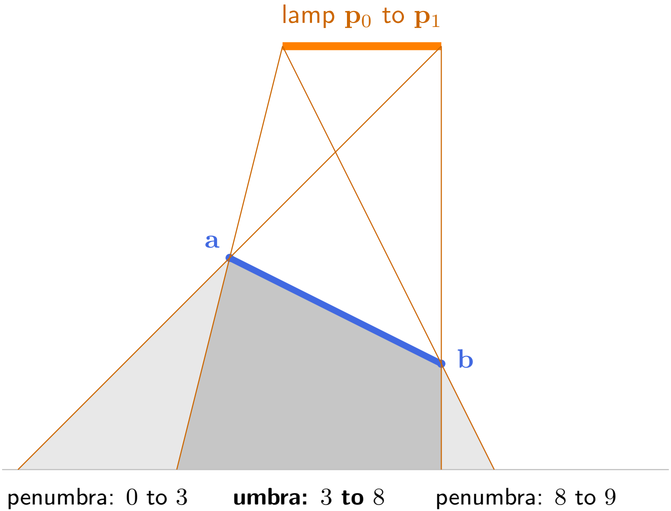
Umbra, penumbra, lit



umbra: 3 **to** 8

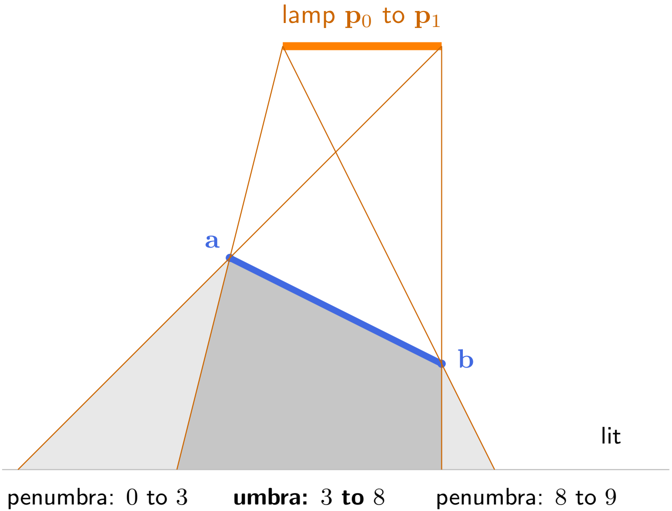
- Umbra: sees none of the lamp

Umbra, penumbra, lit



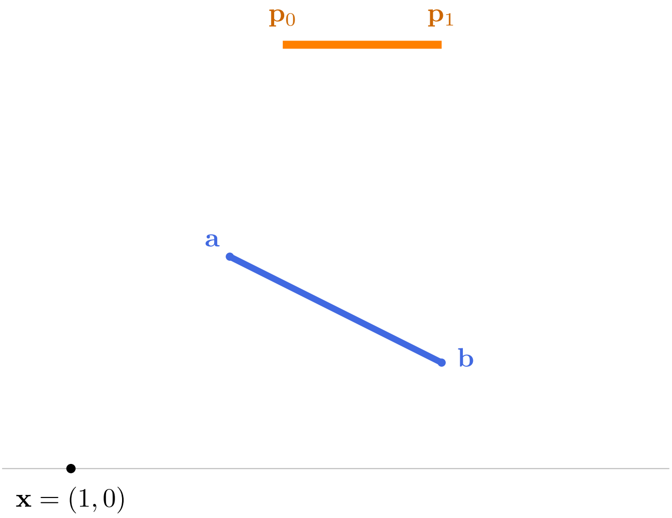
- Umbra: sees none of the lamp
- Penumbra: sees part of it

Umbra, penumbra, lit

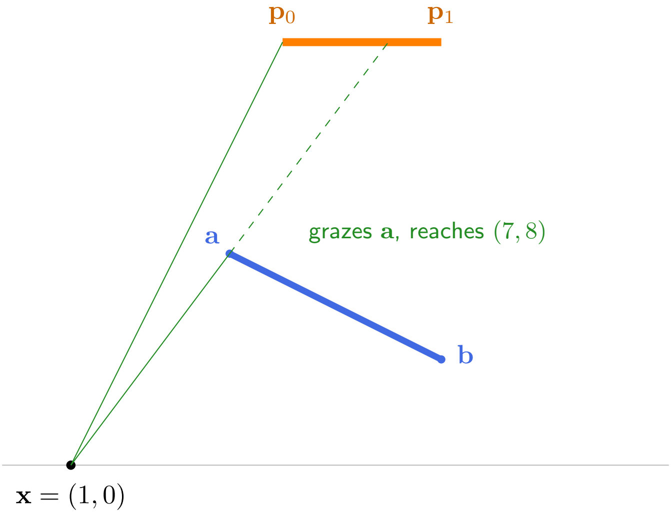


- Umbra: sees none of the lamp
- Penumbra: sees part of it
- Lit: sees all of it

Brightness follows the visible part of the lamp

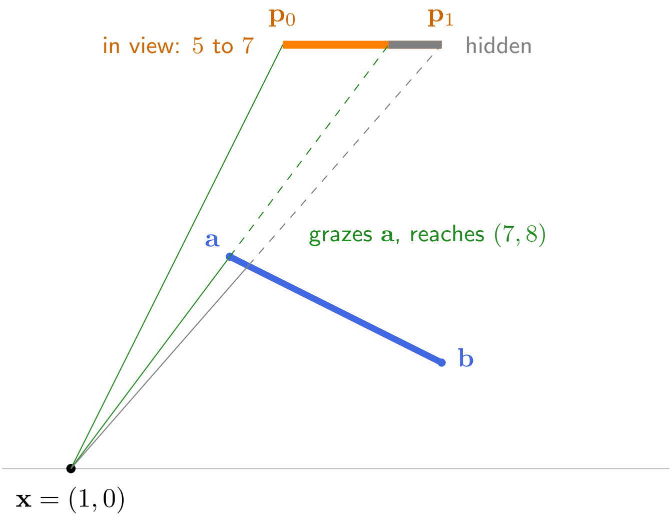


Brightness follows the visible part of the lamp



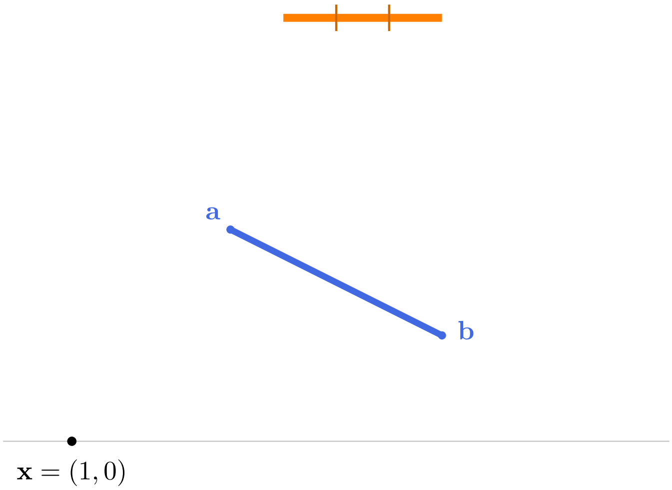
- From $\mathbf{x}$, the plank hides a third

Brightness follows the visible part of the lamp



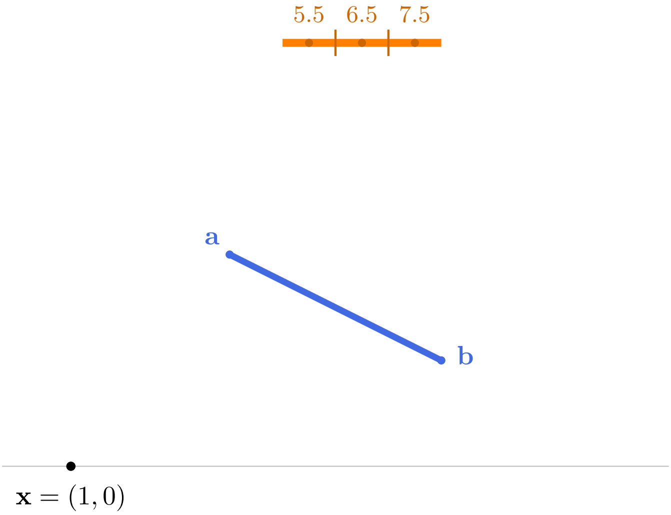
- From $\mathbf{x}$, the plank hides a third
- Two thirds of the lamp in view
- Brightness follows the visible part

One shadow ray per piece of the lamp



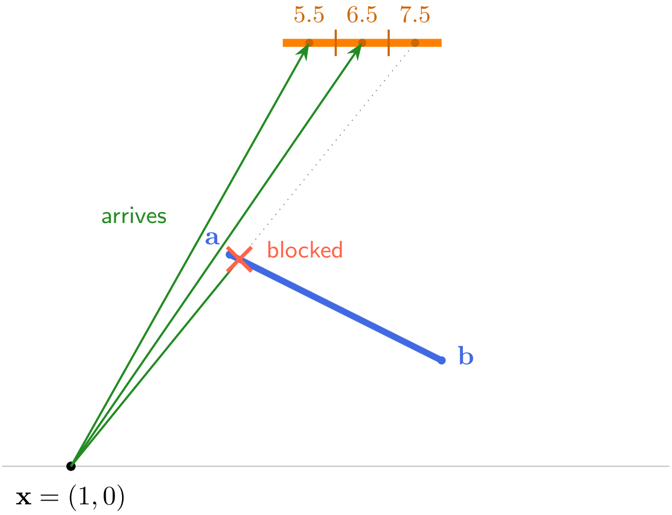
- Split the lamp into thirds

One shadow ray per piece of the lamp



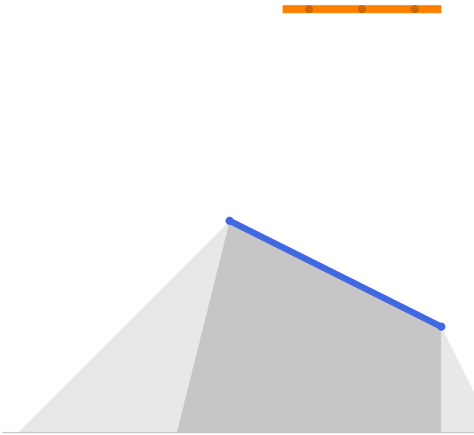
- Split the lamp into thirds
- A sample point at each centre

One shadow ray per piece of the lamp



- Split the lamp into thirds
- A sample point at each centre
- One shadow ray each: two of three arrive

Visibility becomes a fraction

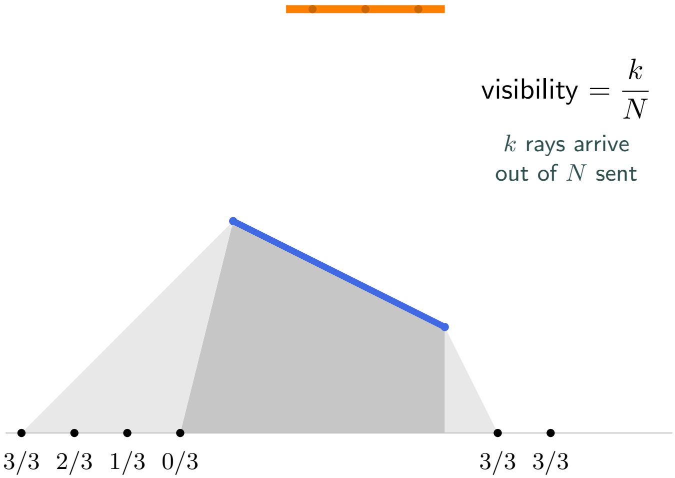


$$\text{visibility} = \frac{k}{N}$$

k rays arrive
out of N sent

- k of N rays arrive

Visibility becomes a fraction

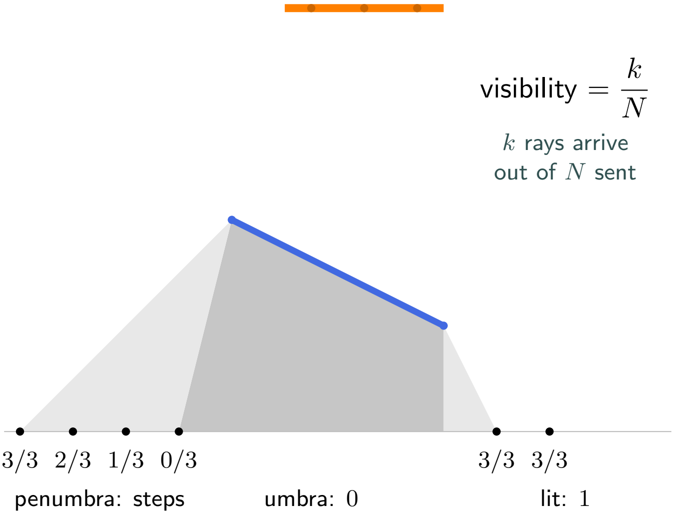


$$\text{visibility} = \frac{k}{N}$$

k rays arrive
out of N sent

- k of N rays arrive
- 1 in the light, 0 in the umbra

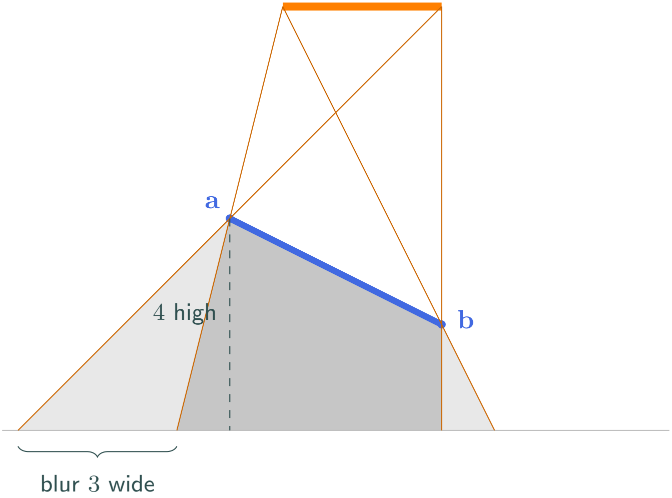
Visibility becomes a fraction



- k of N rays arrive
- 1 in the light, 0 in the umbra
- Steps in between: the penumbra

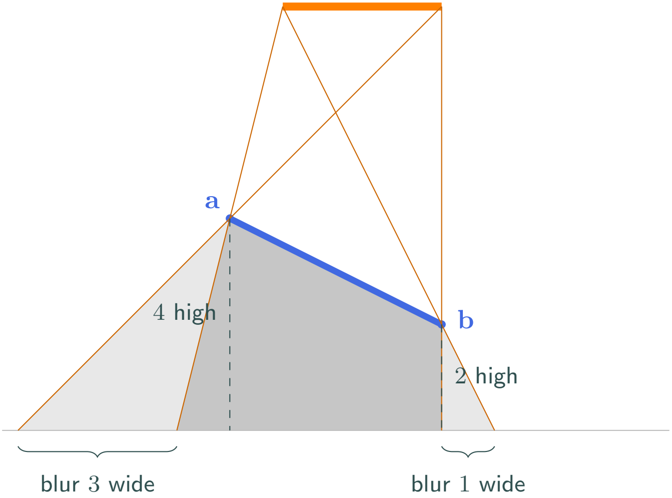
share $=$ visibility $\times$ falloff $\times$ facing

Closer to the floor, sharper the edge



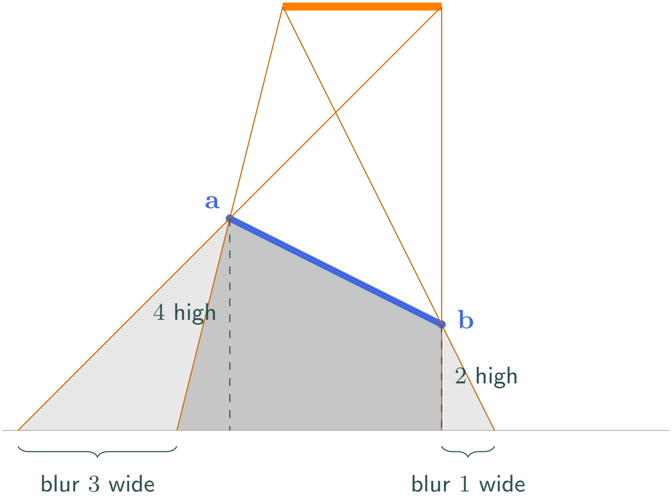
- High end $\mathbf{a}$: blur 3 wide

Closer to the floor, sharper the edge



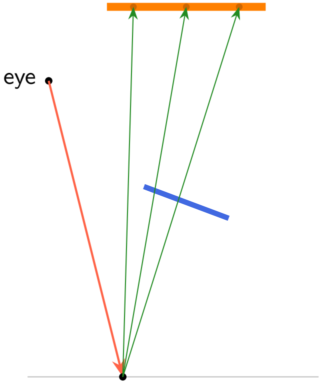
- High end **a**: blur 3 wide
- Low end **b**: blur 1 wide

Closer to the floor, sharper the edge



- High end $\mathbf{a}$: blur 3 wide
- Low end $\mathbf{b}$: blur 1 wide
- On the floor: no blur at all

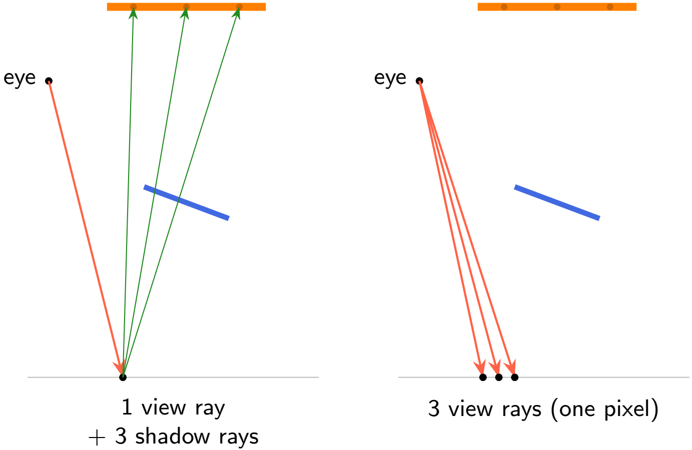
Spread the rays instead of adding them



1 view ray
+ 3 shadow rays

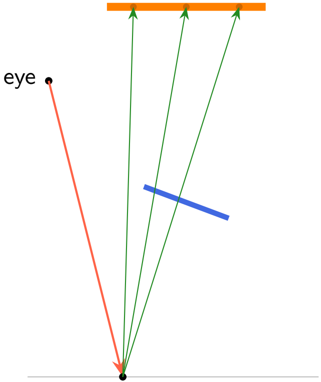
- Three extra rays per point

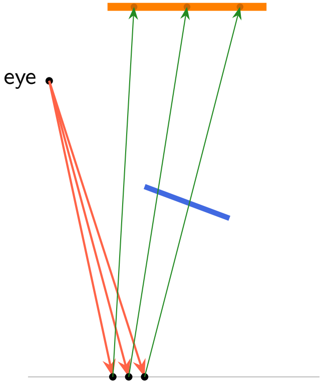
Spread the rays instead of adding them



- Three extra rays per point
- A pixel already sends several rays

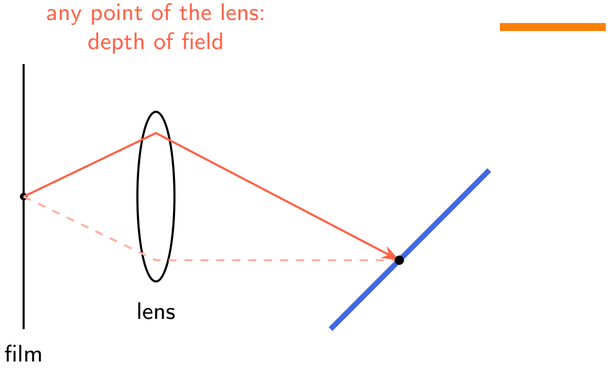
Spread the rays instead of adding them





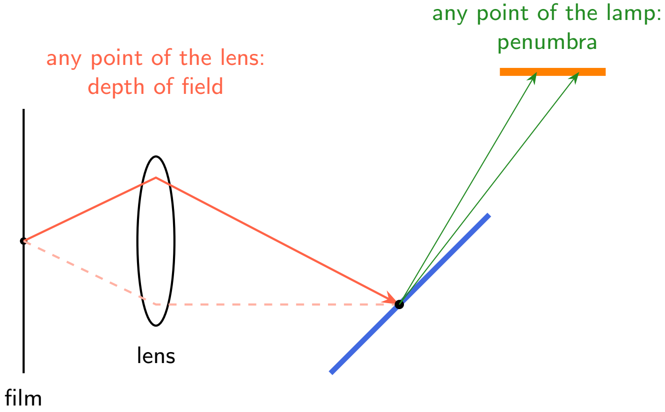
- Three extra rays per point
- A pixel already sends several rays
- Give each a different lamp point: no extra rays

The same trick blurs everything else



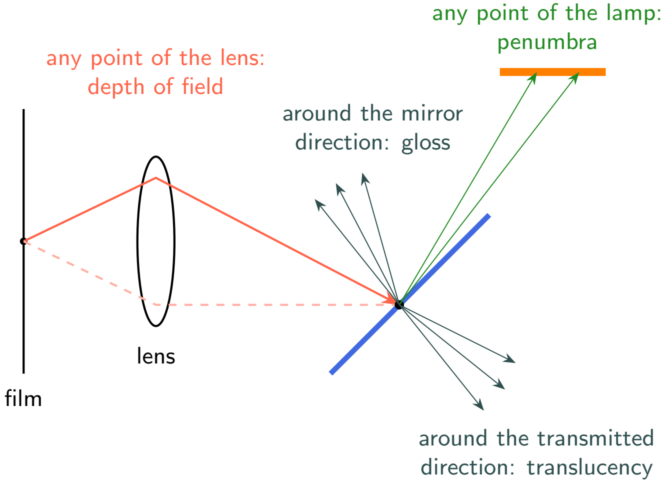
- Lens: depth of field

The same trick blurs everything else



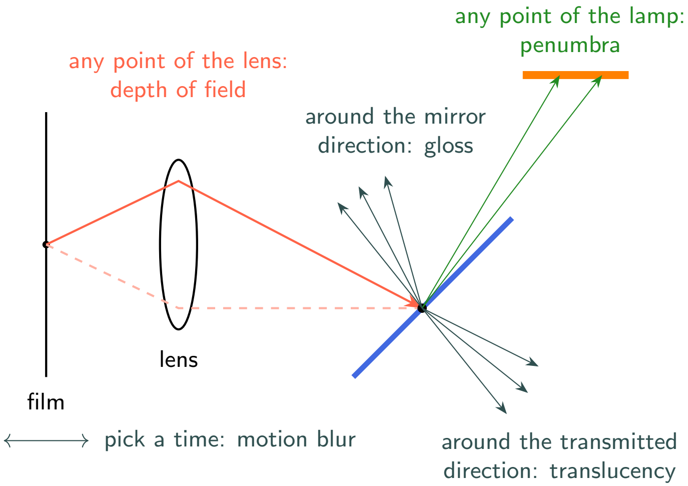
- Lens: depth of field
- Lamp: penumbra

The same trick blurs everything else



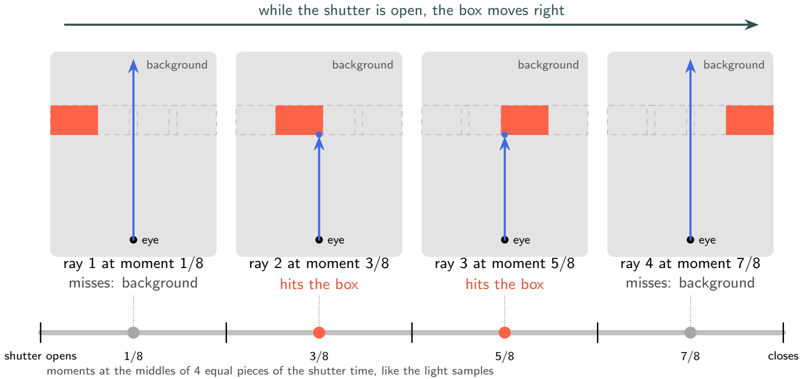
- Lens: depth of field
- Lamp: penumbra
- Reflection: gloss;
  transmission: translucency

The same trick blurs everything else



- Lens: depth of field
- Lamp: penumbra
- Reflection: gloss;
  transmission: translucency
- Time: motion blur

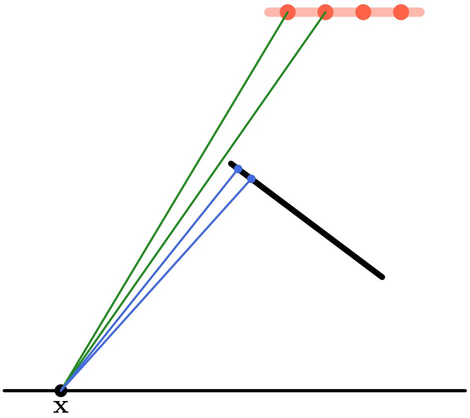
Motion blur: each ray sees its own moment



pixel = average of its 4 rays:  half box, half background

A still box: all 4 rays see the same thing, so the pixel is sharp. A moving box: the rays disagree, the pixel gets a mix, and that mix is the blur.

Recap: soft shadows by sampling the light



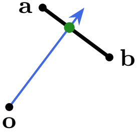
$$f \approx \frac{\text{rays through}}{N}$$

umbra $f = 0$, penumbra $0 < f < 1$, lit $f = 1$

add $f\times$ this light's share

Summary

1 Set the two formulas equal



$$\underbrace{\mathbf{o} + t\,\mathbf{d}}_{\text{ray}} = \underbrace{\mathbf{a} + u\,(\mathbf{b} - \mathbf{a})}_{\text{wall}}$$

2 Cramer's rule gives t and u

$$\underbrace{\begin{bmatrix}\mathbf{d} & \mathbf{a} - \mathbf{b}\end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} t \\ u \end{bmatrix} = \mathbf{a} - \mathbf{o}$$

$$t = |\mathbf{A}_t| \,/\, |\mathbf{A}| \qquad u = |\mathbf{A}_u| \,/\, |\mathbf{A}|$$

$\mathbf{A}_t$, $\mathbf{A}_u$: $\mathbf{A}$ with $\mathbf{a} - \mathbf{o}$ in t's or u's column

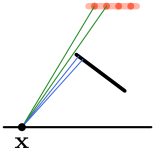
3 Pass all three, or it's a miss

$|\mathbf{A}| \neq 0$ — not parallel

$t \geq 0$ — ahead of $\mathbf{o}$

$0 \leq u \leq 1$ — on the wall

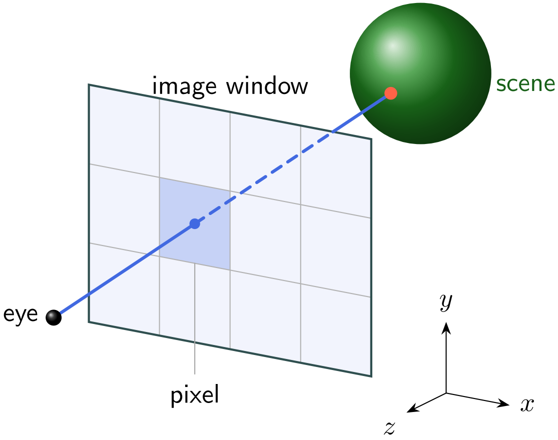
4 Soft shadow: fraction of light seen



$$f \approx \frac{\text{rays through}}{N}$$

N: number of shadow rays

Next lecture: rays leave the flat world



- 3D points and vectors
- Cross product: a perpendicular direction
- A camera: one ray per pixel

Exit Ticket

Today's question *(2 minutes)*

A ray starts at $\mathbf{o} = (1,2)$ with $\mathbf{d} = (1,0)$. A wall runs from $\mathbf{a} = (4,0)$ to $\mathbf{b} = (4,5)$.

(a) Find t and u. Hit or miss?

(b) A floor point fires 8 shadow rays at a light, and 6 are blocked. What is f? Which zone?

CS 453 — Exit Ticket #12

Name: ____________________

Roll #: __________ Date: ________

Answer:

Hand it to me on your way out.
No name, no attendance.