Lecture 8: Homogeneous Coordinates

Composition, Translation, Inverses

CS 453 — Computer Graphics

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Forman Christian University

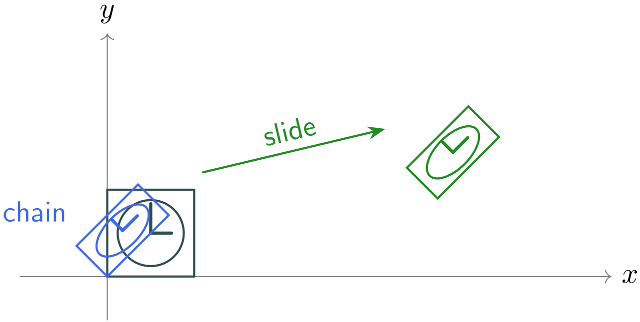
Today: chain moves, slide, and undo

- Same clock as Lecture 7

Today: chain moves, slide, and undo

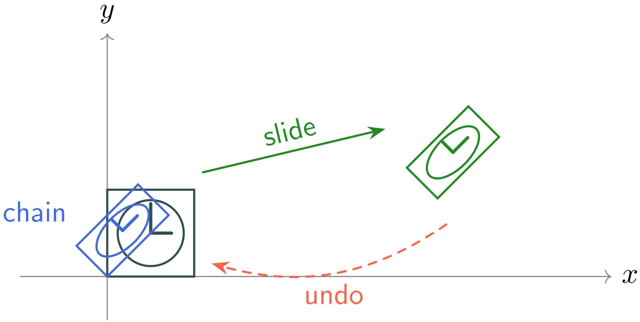
- Same clock as Lecture 7
- Chain two moves into one

Today: chain moves, slide, and undo



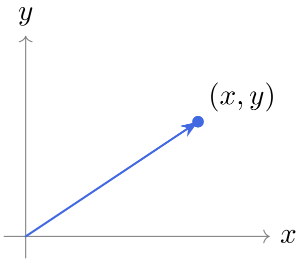
- Same clock as Lecture 7
- Chain two moves into one
- Slide it: a new trick

Today: chain moves, slide, and undo

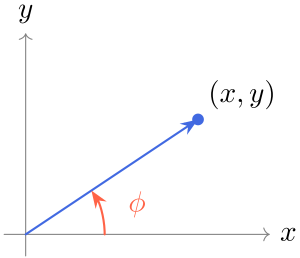


- Same clock as Lecture 7
- Chain two moves into one
- Slide it: a new trick
- Undo any chain

Our conventions in one picture



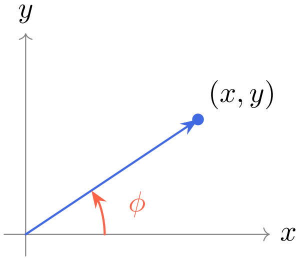
Our conventions in one picture



$\phi > 0$: counterclockwise

- Positive angle: counterclockwise

Our conventions in one picture



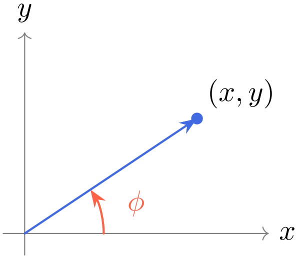
$\phi > 0$: counterclockwise

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

column vectors, matrix on the left

- Positive angle: counterclockwise
- Column vector, matrix on its left

Our conventions in one picture



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$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

column vectors, matrix on the left

$$\mathbf{M} = \mathbf{R} \ \mathbf{S}$$

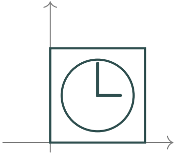
$\mathbf{S}$ first, then $\mathbf{R}$

- Positive angle: counterclockwise
- Column vector, matrix on its left
- Read products right to left

Composition

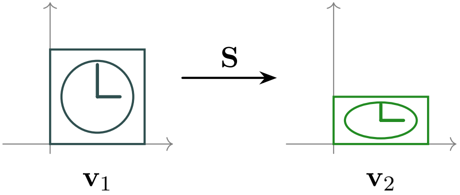
Many moves, one matrix

Two moves collapse into one matrix



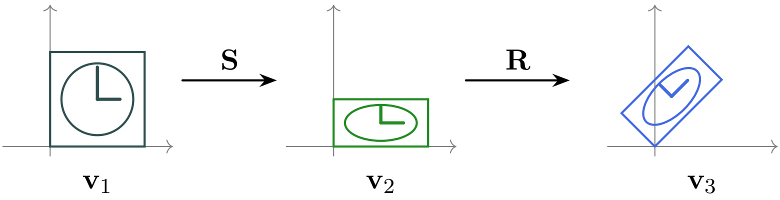
$\mathbf{v}_1$

Two moves collapse into one matrix



$\mathbf{v}_2 = \mathbf{S}\mathbf{v}_1$ first

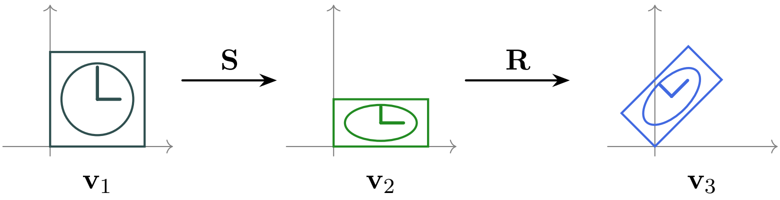
Two moves collapse into one matrix



$\mathbf{v}_2 = \mathbf{S}\mathbf{v}_1$ first

$\mathbf{v}_3 = \mathbf{R}\mathbf{v}_2$ then

Two moves collapse into one matrix

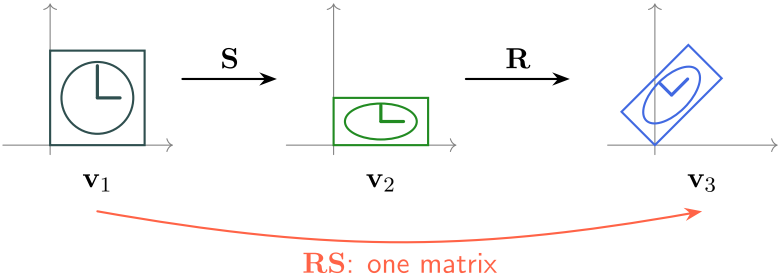


$\mathbf{v}_2 = \mathbf{S}\mathbf{v}_1$ first

$\mathbf{v}_3 = \mathbf{R}\mathbf{v}_2$ then

$\mathbf{v}_3 = \mathbf{R}(\mathbf{S}\mathbf{v}_1)$ substitute $\mathbf{v}_2$

Two moves collapse into one matrix



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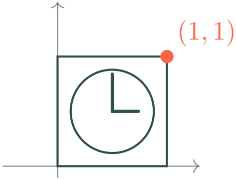
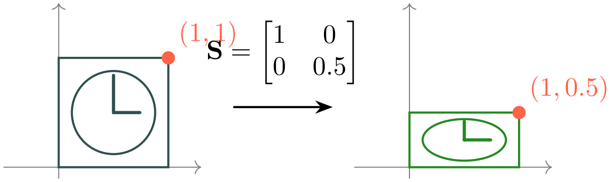
$\mathbf{v}_3 = \mathbf{R}(\mathbf{S}\mathbf{v}_1)$ substitute $\mathbf{v}_2$

$\mathbf{v}_3 = (\mathbf{RS})\,\mathbf{v}_1$ associative

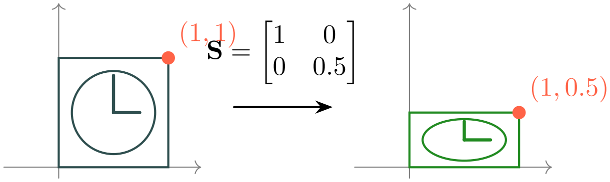
Example 12: squash, then turn 45°



Example 12: squash, then turn 45°



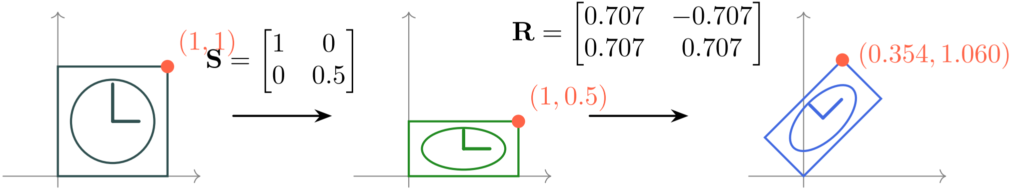
Example 12: squash, then turn 45°



$$\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\mathbf{S}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 1 \\ 0 \cdot 1 + 0.5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

Example 12: squash, then turn 45°



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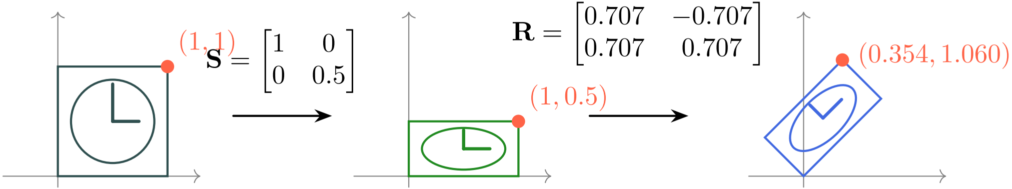
Example 12: squash, then turn 45°



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$$\mathbf{R}\begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.707 \cdot 1 + (-0.707) \cdot 0.5 \\ 0.707 \cdot 1 + 0.707 \cdot 0.5 \end{bmatrix} = \begin{bmatrix} 0.707 - 0.353 \\ 0.707 + 0.353 \end{bmatrix} = \begin{bmatrix} 0.354 \\ 1.060 \end{bmatrix}$$

Multiply $\mathbf{R}$ times $\mathbf{S}$, one entry at a time

$$\mathbf{RS} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

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$$p_{11} = 0.707 \cdot 1 + (-0.707) \cdot 0 = 0.707$$ row 1, column 1

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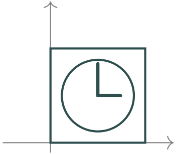
$$\mathbf{RS} = \begin{bmatrix} 0.707 & -0.353 \\ 0.707 & 0.353 \end{bmatrix}$$

One matrix gives the same clock



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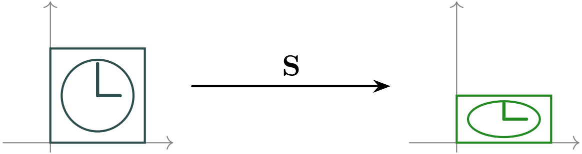

- Two steps: $\mathbf{S}$, then $\mathbf{R}$

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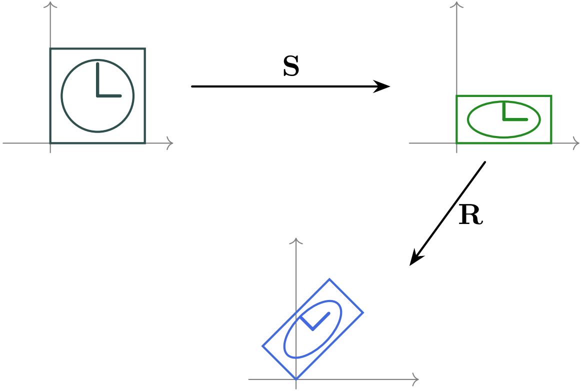

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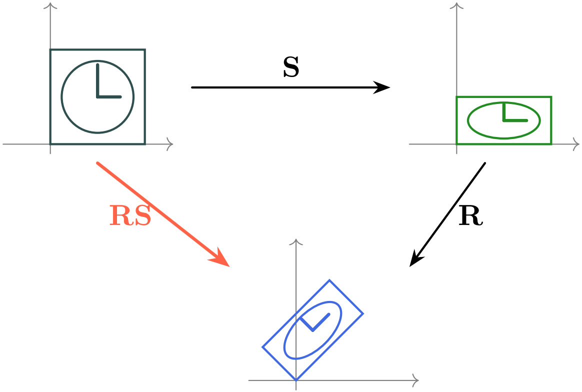
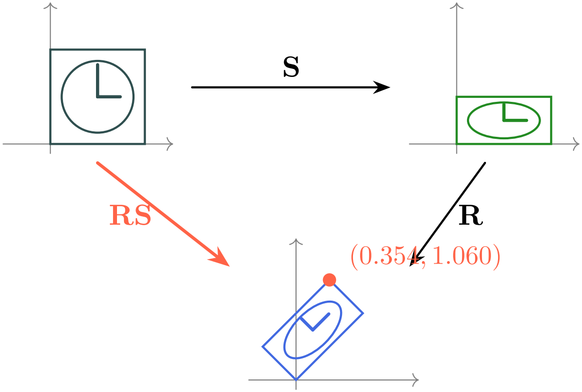


- Two steps: $\mathbf{S}$, then $\mathbf{R}$
- One step: the product $\mathbf{RS}$

One matrix gives the same clock



- Two steps: $\mathbf{S}$, then $\mathbf{R}$
- One step: the product $\mathbf{RS}$
- Same corner either way

$$\begin{bmatrix} 0.707 & -0.353 \\ 0.707 & 0.353 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.707 - 0.353 \\ 0.707 + 0.353 \end{bmatrix}$$

Swap the order: $\mathbf{SR}$ is a new matrix

$$\mathbf{SR} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

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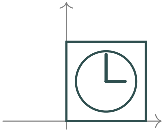
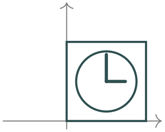
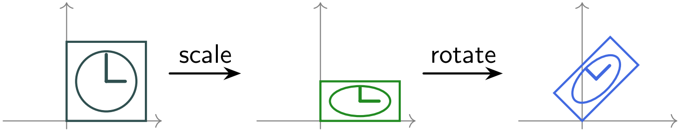
$$\mathbf{SR} = \begin{bmatrix} 0.707 & -0.707 \\ 0.353 & 0.353 \end{bmatrix} \neq \begin{bmatrix} 0.707 & -0.353 \\ 0.707 & 0.353 \end{bmatrix} = \mathbf{RS}$$

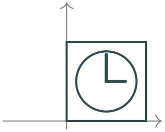
Swap the order, get a different clock





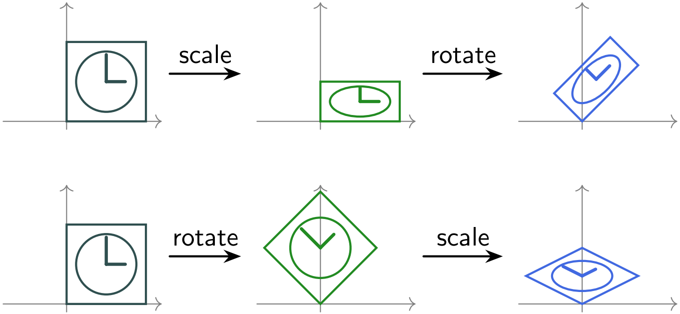
Swap the order, get a different clock





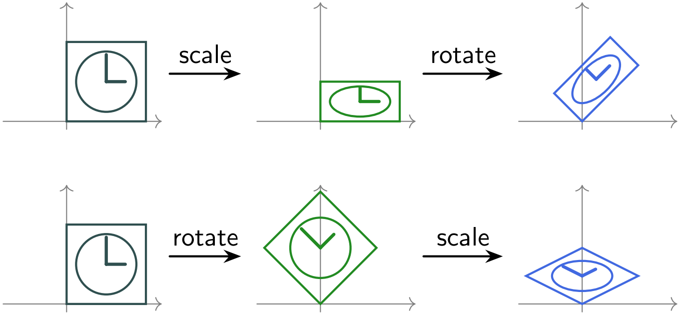
- Top: scale, then rotate

Swap the order, get a different clock



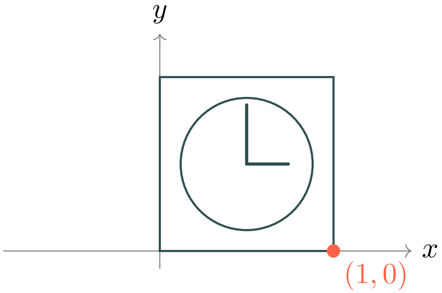
- Top: scale, then rotate
- Bottom: rotate, then scale

Swap the order, get a different clock



- Top: scale, then rotate
- Bottom: rotate, then scale
- Same two moves, different result

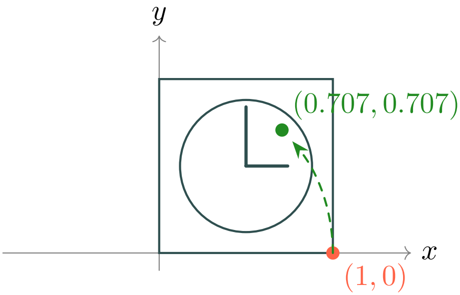
Quick check 1: which move acts first?



$$\mathbf{M} = \mathbf{SR}$$

Which acts first?
Where does $(1, 0)$ land?

Quick check 1: which move acts first?



$$\mathbf{M} = \mathbf{S}\mathbf{R}$$

Which acts first?
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$\mathbf{R}$ first: $\begin{bmatrix} 0.707 \cdot 1 + (-0.707) \cdot 0 \\ 0.707 \cdot 1 + 0.707 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$

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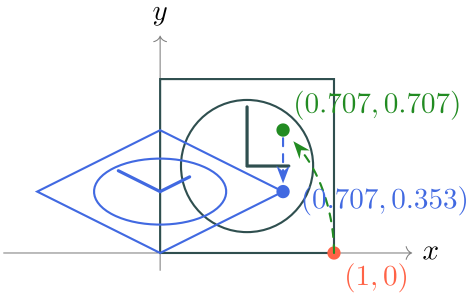


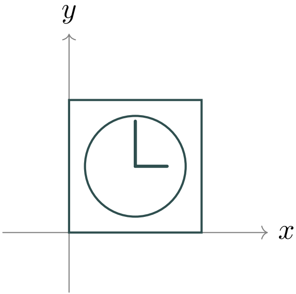
$$\mathbf{M} = \mathbf{SR}$$

Which acts first?
Where does $(1, 0)$ land?

R first: $\begin{bmatrix} 0.707 \cdot 1 + (-0.707) \cdot 0 \\ 0.707 \cdot 1 + 0.707 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$

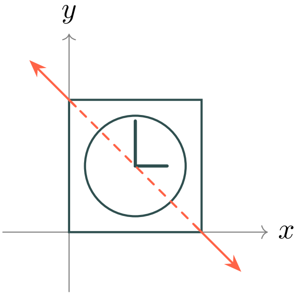
then **S**: $\begin{bmatrix} 1 \cdot 0.707 + 0 \cdot 0.707 \\ 0 \cdot 0.707 + 0.5 \cdot 0.707 \end{bmatrix} = \begin{bmatrix} 0.707 \\ 0.353 \end{bmatrix}$

Example 13: stretch along a diagonal



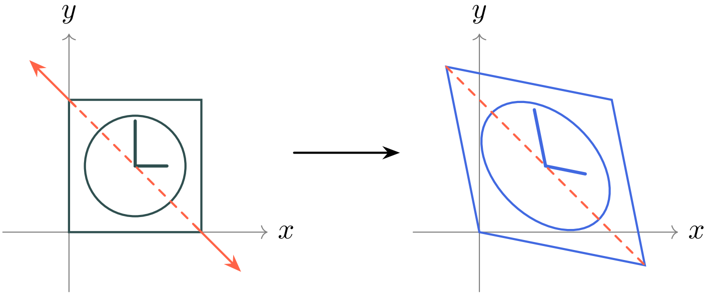
- Scale stretches only along axes

Example 13: stretch along a diagonal



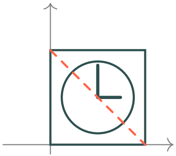
- Scale stretches only along axes
- Goal: stretch along this diagonal

Example 13: stretch along a diagonal

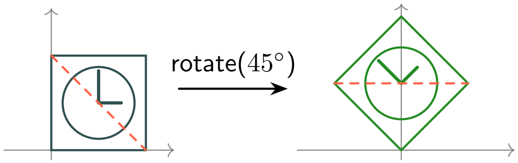


- Scale stretches only along axes
- Goal: stretch along this diagonal
- Result: 50% longer that way

Turn, stretch, turn back

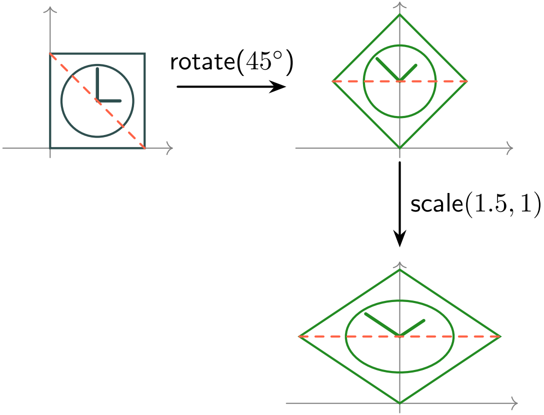


Turn, stretch, turn back



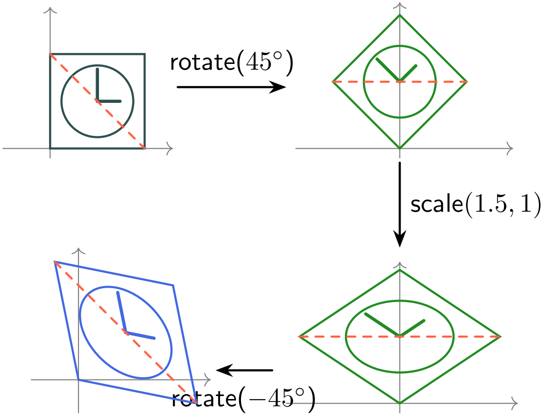
- Turn: diagonal now lies flat

Turn, stretch, turn back



- Turn: diagonal now lies flat
- Stretch along x by 1.5

Turn, stretch, turn back



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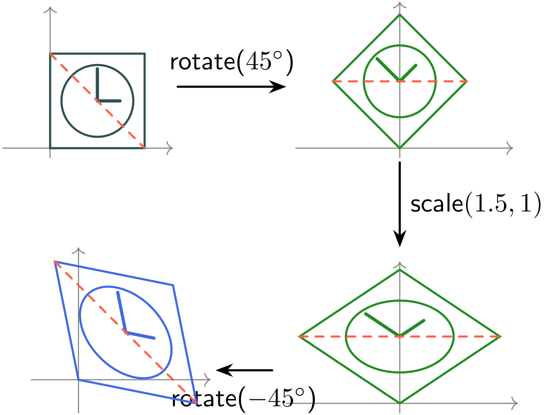

- Turn: diagonal now lies flat
- Stretch along x by 1.5
- Turn back
- Read right to left

$$\mathrm{rotate}(-45^\circ)\,\mathrm{scale}(1.5,1)\,\mathrm{rotate}(45^\circ)$$

Multiply the three: a symmetric matrix

$\mathbf{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{R}^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

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$$\mathbf{RSR}^{\mathrm{T}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

Multiply the three: a symmetric matrix

$\mathbf{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $\mathbf{S} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathbf{R}^{\mathrm{T}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\mathbf{RSR}^{\mathrm{T}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 \cdot 1 + 0 \cdot 1 & 1.5 \cdot (-1) + 0 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot (-1) + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & -1.5 \\ 1 & 1 \end{bmatrix}$$ right pair first

Multiply the three: a symmetric matrix

$$\mathbf{R} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},\ \mathbf{S} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix},\ \mathbf{R}^{\mathrm{T}} =$$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{RSR}^{\mathrm{T}} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1.5 \cdot 1 + 0 \cdot 1 & 1.5 \cdot (-1) + 0 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot (-1) + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & -1.5 \\ 1 & 1 \end{bmatrix}$$ right pair first

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1.5 & -1.5 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1.5 + 1 \cdot 1 & 1 \cdot (-1.5) + 1 \cdot 1 \\ (-1) \cdot 1.5 + 1 \cdot 1 & (-1)(-1.5) + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 & -0.5 \\ -0.5 & 2.5 \end{bmatrix}$$ then the left

Multiply the three: a symmetric matrix

$$\mathbf{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \ \mathbf{S} = \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{R}^{\mathrm{T}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{RSR}^{\mathrm{T}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$$

$$\begin{bmatrix} 1.5 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1.5 \cdot 1 + 0 \cdot 1 & 1.5 \cdot (-1) + 0 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot (-1) + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.5 & -1.5 \\ 1 & 1 \end{bmatrix}$$

right pair first

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1.5 & -1.5 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot 1.5 + 1 \cdot 1 & 1 \cdot (-1.5) + 1 \cdot 1 \\ (-1) \cdot 1.5 + 1 \cdot 1 & (-1)(-1.5) + 1 \cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5 & -0.5 \\ -0.5 & 2.5 \end{bmatrix}$$

then the left

$$\mathbf{RSR}^{\mathrm{T}} = \frac{1}{2} \begin{bmatrix} 2.5 & -0.5 \\ -0.5 & 2.5 \end{bmatrix} = \begin{bmatrix} 1.25 & -0.25 \\ -0.25 & 1.25 \end{bmatrix}$$

symmetric

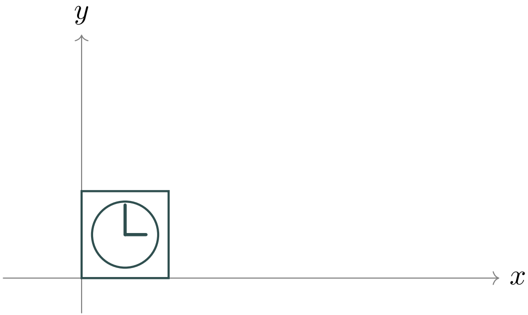
Recap: composition

$$\mathbf{v}_3 = \mathbf{R}(\mathbf{S}\mathbf{v}_1) = (\mathbf{R}\mathbf{S})\,\mathbf{v}_1$$

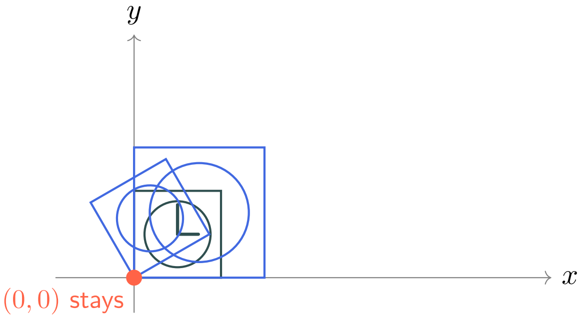
Translation and Homogeneous Coordinates

Sliding with one matrix

No 2×2 matrix can slide the clock



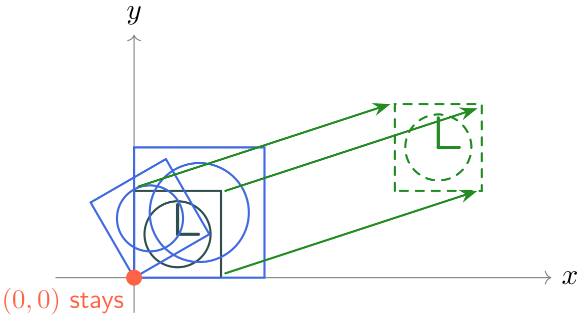
No 2×2 matrix can slide the clock



- Scale, rotate: origin stays put

$$x' = m_{11}x + m_{12}y$$

No 2×2 matrix can slide the clock



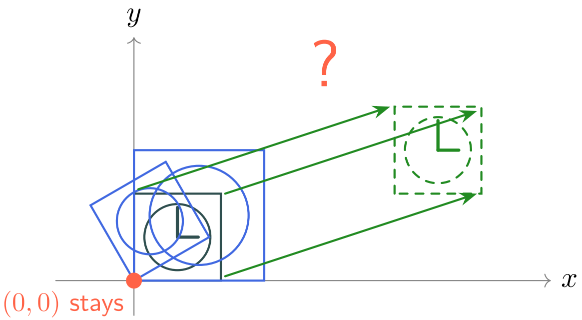
- Scale, rotate: origin stays put

  $$x' = m_{11}x + m_{12}y$$

- Slide: every point moves equally

  $$x' = x + x_t, \quad y' = y + y_t$$

No 2×2 matrix can slide the clock



- Scale, rotate: origin stays put

  $$x' = m_{11}x + m_{12}y$$

- Slide: every point moves equally

  $$x' = x + x_t, \quad y' = y + y_t$$

- No 2×2 matrix does that

Add a third coordinate, always 1

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

- Point (x, y) gets a 1

$$(x, y) \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Add a third coordinate, always 1

scale / rotate part

$$\begin{bmatrix} \boxed{\begin{matrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{matrix}} & \begin{matrix} x_t \\ y_t \end{matrix} \\ \begin{matrix} 0 & \quad 0 \end{matrix} & 1 \end{bmatrix}$$

- Point (x, y) gets a 1

$$(x, y) \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Upper left: scale and rotate

Add a third coordinate, always 1

scale / rotate part

$$\begin{bmatrix} \boxed{\begin{matrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{matrix}} & \boxed{\begin{matrix} x_t \\ y_t \end{matrix}} \\ 0 \quad 0 & 1 \end{bmatrix}$$

translation

- Point (x, y) gets a 1

$$(x, y) \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Upper left: scale and rotate
- Last column: the translation

Add a third coordinate, always 1

scale / rotate part

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

translation

fixed row: copies the 1

- Point (x, y) gets a 1

$$(x, y) \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- Upper left: scale and rotate
- Last column: the translation
- Bottom row stays $0\ 0\ 1$

Multiply out: linear part plus shift

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Multiply out: linear part plus shift

$$\begin{bmatrix} \boxed{m_{11} \quad m_{12} \quad x_t} \\ m_{21} \quad m_{22} \quad y_t \\ 0 \quad 0 \quad 1 \end{bmatrix} \boxed{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}$$

$$x' = m_{11}x + m_{12}y + x_t \cdot 1 \qquad \text{row 1}$$

Multiply out: linear part plus shift

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ \boxed{m_{21} \quad m_{22} \quad y_t} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$x' = m_{11}x + m_{12}y + x_t \cdot 1$ row 1

$y' = m_{21}x + m_{22}y + y_t \cdot 1$ row 2

Multiply out: linear part plus shift

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ \boxed{0 \quad 0 \quad 1} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$x' = m_{11}x + m_{12}y + x_t \cdot 1$ row 1

$y' = m_{21}x + m_{22}y + y_t \cdot 1$ row 2

$0 \cdot x + 0 \cdot y + 1 \cdot 1 = 1$ row 3 copies 1

Multiply out: linear part plus shift

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

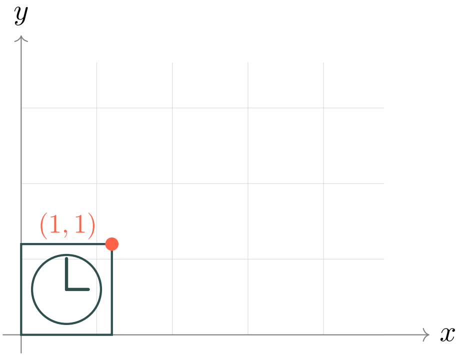
$x' = m_{11}x + m_{12}y + x_t \cdot 1$ row 1

$y' = m_{21}x + m_{22}y + y_t \cdot 1$ row 2

$0 \cdot x + 0 \cdot y + 1 \cdot 1 = 1$ row 3 copies 1

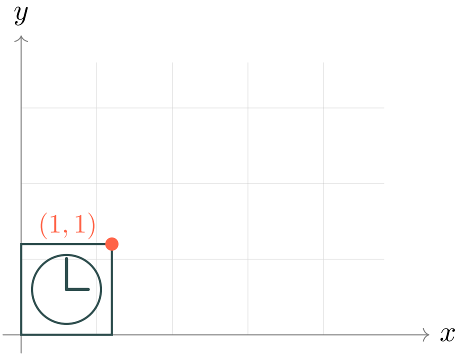
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11}x + m_{12}y + x_t \\ m_{21}x + m_{22}y + y_t \\ 1 \end{bmatrix}$$

Slide the clock by $(2, 1)$



$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

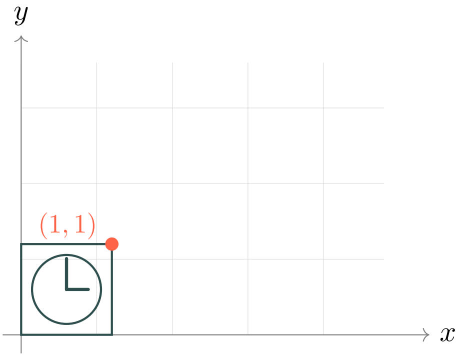
Slide the clock by $(2, 1)$



$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x' = 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 = 3$$

Slide the clock by $(2, 1)$

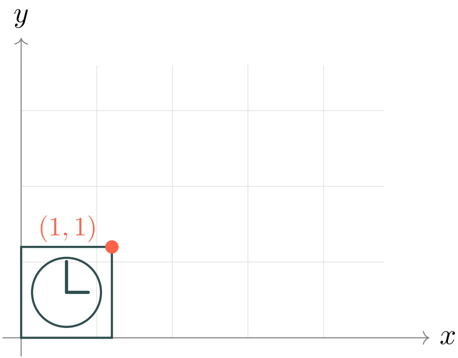


$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x' = 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 = 3$$

$$y' = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2$$

Slide the clock by $(2, 1)$



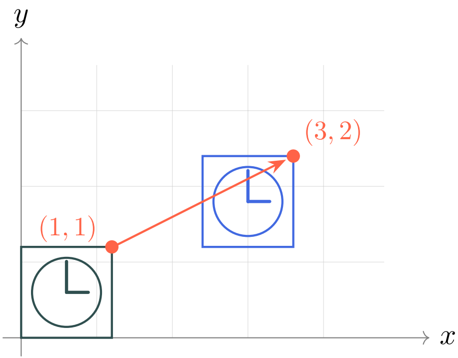
$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$x' = 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 = 3$$

$$y' = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2$$

$$0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 = 1$$

Slide the clock by $(2, 1)$



$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

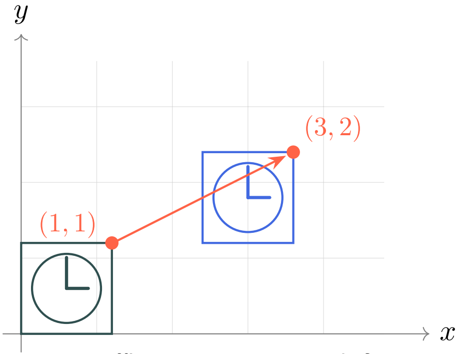
$$x' = 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 = 3$$

$$y' = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2$$

$$0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 = 1$$

- Every corner slides by $(2, 1)$

Slide the clock by $(2, 1)$



affine: linear part + shift

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

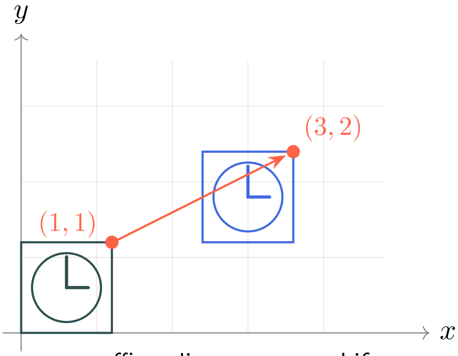
$$x' = 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 = 3$$

$$y' = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2$$

$$0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 = 1$$

- Every corner slides by $(2, 1)$
- Name: affine transformation

Slide the clock by $(2, 1)$



affine: linear part + shift

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

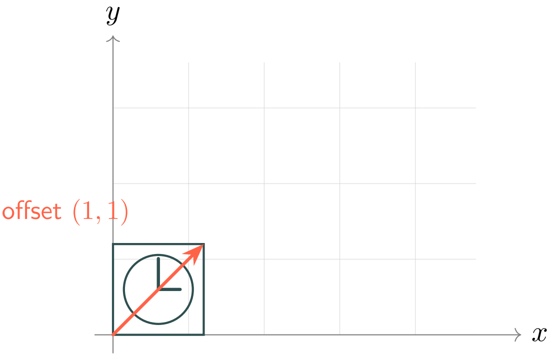
$x' = 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 = 3$

$y' = 0 \cdot 1 + 1 \cdot 1 + 1 \cdot 1 = 2$

$0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 = 1$

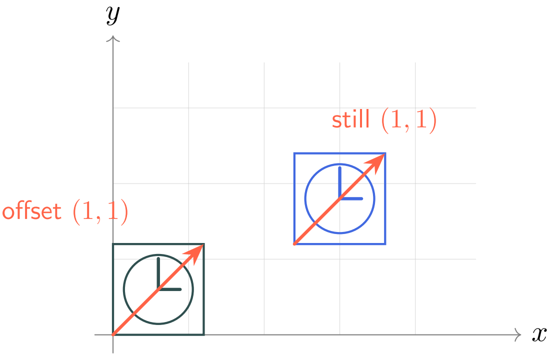
- Every corner slides by $(2, 1)$
- Name: affine transformation
- Extra 1: homogeneous coordinates

Directions get a 0 and ignore shifts



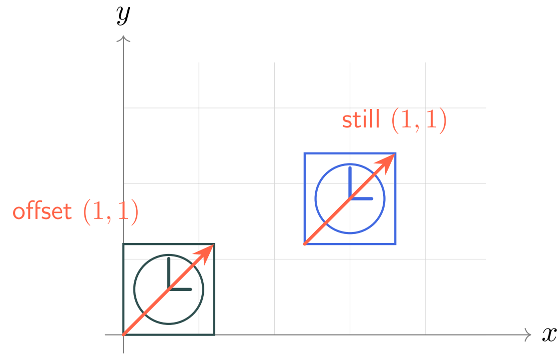
- Offset from corner to corner

Directions get a 0 and ignore shifts



- Offset from corner to corner
- Clock slides, offset does not

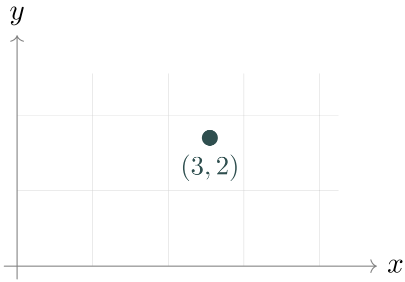
Directions get a 0 and ignore shifts



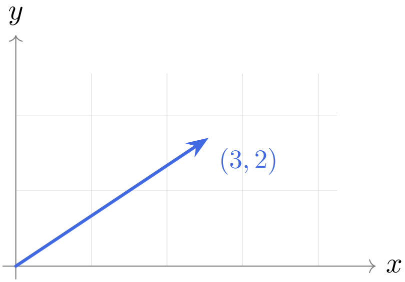
- Offset from corner to corner
- Clock slides, offset does not
- Third entry 0: shift ignored

$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Translate by $(2, 1)$: places move, arrows don't

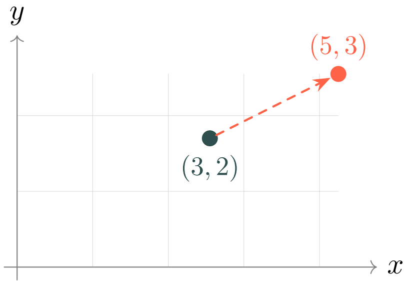


Last entry 1: a location



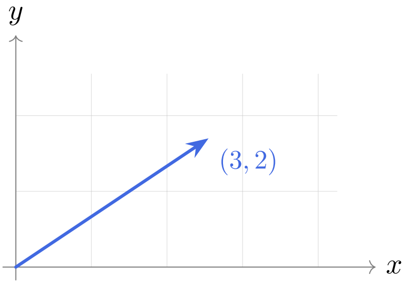
Last entry 0: a direction

Translate by $(2,1)$: places move, arrows don't



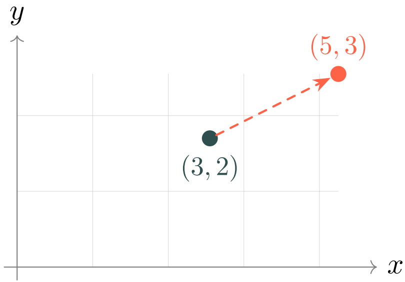
$$\begin{bmatrix} 1\cdot 3+0\cdot 2+2\cdot 1 \\ 0\cdot 3+1\cdot 2+1\cdot 1 \\ 0\cdot 3+0\cdot 2+1\cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

Last entry 1: a location



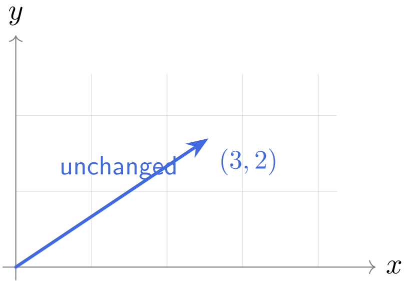
Last entry 0: a direction

Translate by $(2,1)$: places move, arrows don't



$$\begin{bmatrix} 1\cdot 3+0\cdot 2+2\cdot 1 \\ 0\cdot 3+1\cdot 2+1\cdot 1 \\ 0\cdot 3+0\cdot 2+1\cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

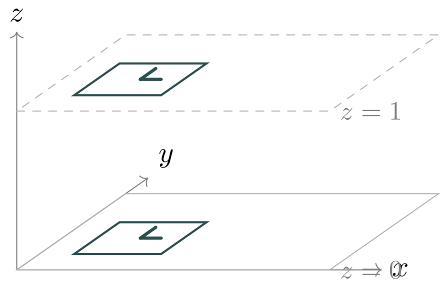
Last entry 1: a location



$$\begin{bmatrix} 1\cdot 3+0\cdot 2+2\cdot 0 \\ 0\cdot 3+1\cdot 2+1\cdot 0 \\ 0\cdot 3+0\cdot 2+1\cdot 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

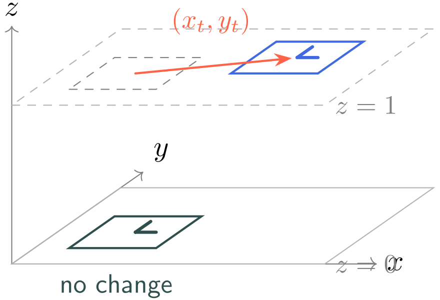
Last entry 0: a direction

Another view: a 3D shear



$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + x_t z \\ y + y_t z \\ z \end{bmatrix}$$

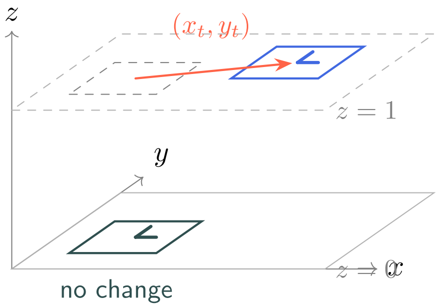
Another view: a 3D shear



$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + x_t z \\ y + y_t z \\ z \end{bmatrix}$$

- Each layer slides by its height

Another view: a 3D shear

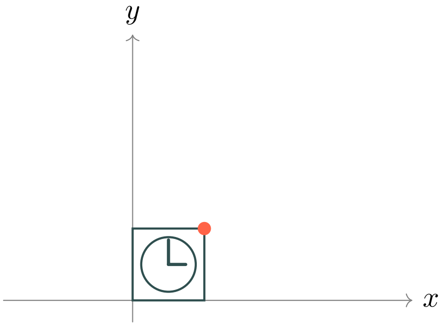


$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + x_t z \\ y + y_t z \\ z \end{bmatrix}$$

- Each layer slides by its height
- Layer $z = 1$: our 2D slide

$$z = 1 : \quad \begin{bmatrix} x + x_t \\ y + y_t \\ 1 \end{bmatrix}$$

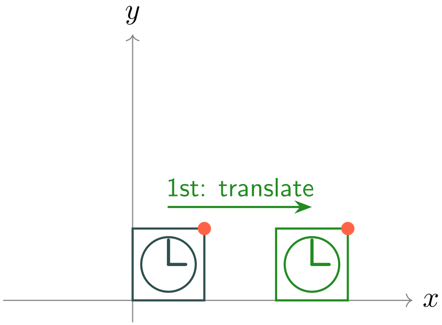
Translate, then rotate: one matrix $\mathbf{M}$



$$\mathbf{M} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{rotate}} \underbrace{\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix}}_{\text{translate}}$$

- Right matrix acts first

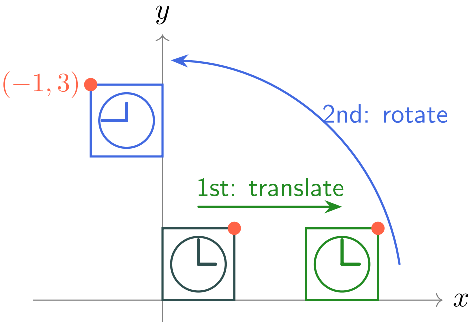
Translate, then rotate: one matrix $\mathbf{M}$



$$\mathbf{M} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{rotate}} \underbrace{\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix}}_{\text{translate}}$$

- Right matrix acts first

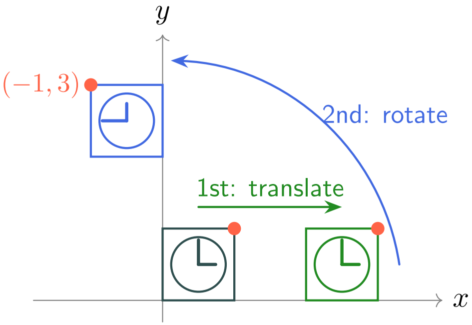
Translate, then rotate: one matrix $\mathbf{M}$



$$\mathbf{M} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{rotate}} \underbrace{\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix}}_{\text{translate}}$$

- Right matrix acts first

Translate, then rotate: one matrix $\mathbf{M}$



$$\mathbf{M} = \underbrace{\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\text{rotate}} \underbrace{\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix}}_{\text{translate}}$$

$$\phi = 90^\circ,\ (x_t, y_t) = (2, 0) : \quad \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Right matrix acts first
- Rotation: zeros in shift slots

Multiply out $\mathbf{M}$, one row at a time

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply out $\mathbf{M}$, one row at a time

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(0 \cdot 1 + (-1) \cdot 0 + 0 \cdot 0,\;\; 0 \cdot 0 + (-1) \cdot 1 + 0 \cdot 0,\;\; 0 \cdot 2 + (-1) \cdot 0 + 0 \cdot 1\big) = (0, -1, 0)$

Multiply out $\mathbf{M}$, one row at a time

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(0 \cdot 1 + (-1) \cdot 0 + 0 \cdot 0,\ \ 0 \cdot 0 + (-1) \cdot 1 + 0 \cdot 0,\ \ 0 \cdot 2 + (-1) \cdot 0 + 0 \cdot 1\big) = (0, -1, 0)$

row 2: $\big(1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0,\ \ 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0,\ \ 1 \cdot 2 + 0 \cdot 0 + 0 \cdot 1\big) = (1, 0, 2)$

Multiply out $\mathbf{M}$, one row at a time

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(0 \cdot 1 + (-1) \cdot 0 + 0 \cdot 0,\;\; 0 \cdot 0 + (-1) \cdot 1 + 0 \cdot 0,\;\; 0 \cdot 2 + (-1) \cdot 0 + 0 \cdot 1\big) = (0, -1, 0)$

row 2: $\big(1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0,\;\; 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0,\;\; 1 \cdot 2 + 0 \cdot 0 + 0 \cdot 1\big) = (1, 0, 2)$

row 3: $\big(0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0,\;\; 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0,\;\; 0 \cdot 2 + 0 \cdot 0 + 1 \cdot 1\big) = (0, 0, 1)$

Multiply out $\mathbf{M}$, one row at a time

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(0 \cdot 1 + (-1) \cdot 0 + 0 \cdot 0,\;\; 0 \cdot 0 + (-1) \cdot 1 + 0 \cdot 0,\;\; 0 \cdot 2 + (-1) \cdot 0 + 0 \cdot 1\big) = (0, -1, 0)$

row 2: $\big(1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0,\;\; 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0,\;\; 1 \cdot 2 + 0 \cdot 0 + 0 \cdot 1\big) = (1, 0, 2)$

row 3: $\big(0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0,\;\; 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0,\;\; 0 \cdot 2 + 0 \cdot 0 + 1 \cdot 1\big) = (0, 0, 1)$

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply out $\mathbf{M}$, one row at a time

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(0 \cdot 1 + (-1) \cdot 0 + 0 \cdot 0,\ \ 0 \cdot 0 + (-1) \cdot 1 + 0 \cdot 0,\ \ 0 \cdot 2 + (-1) \cdot 0 + 0 \cdot 1\big) = (0, -1, 0)$

row 2: $\big(1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0,\ \ 1 \cdot 0 + 0 \cdot 1 + 0 \cdot 0,\ \ 1 \cdot 2 + 0 \cdot 0 + 0 \cdot 1\big) = (1, 0, 2)$

row 3: $\big(0 \cdot 1 + 0 \cdot 0 + 1 \cdot 0,\ \ 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 0,\ \ 0 \cdot 2 + 0 \cdot 0 + 1 \cdot 1\big) = (0, 0, 1)$

$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{M} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 1 + (-1) \cdot 1 + 0 \cdot 1 \\ 1 \cdot 1 + 0 \cdot 1 + 2 \cdot 1 \\ 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

Shift second: the two parts stay apart

$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Shift second: the two parts stay apart

$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(1a_{11} + 0a_{21} + x_t \cdot 0,\;\; 1a_{12} + 0a_{22} + x_t \cdot 0,\;\; 1 \cdot 0 + 0 \cdot 0 + x_t \cdot 1\big) = (a_{11}, a_{12}, x_t)$

Shift second: the two parts stay apart

$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\left(1a_{11} + 0a_{21} + x_t \cdot 0,\;\; 1a_{12} + 0a_{22} + x_t \cdot 0,\;\; 1 \cdot 0 + 0 \cdot 0 + x_t \cdot 1\right) = (a_{11}, a_{12}, x_t)$

row 2: $\left(0a_{11} + 1a_{21} + y_t \cdot 0,\;\; 0a_{12} + 1a_{22} + y_t \cdot 0,\;\; 0 \cdot 0 + 1 \cdot 0 + y_t \cdot 1\right) = (a_{21}, a_{22}, y_t)$

Shift second: the two parts stay apart

$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(1a_{11} + 0a_{21} + x_t \cdot 0,\;\; 1a_{12} + 0a_{22} + x_t \cdot 0,\;\; 1 \cdot 0 + 0 \cdot 0 + x_t \cdot 1\big) = (a_{11}, a_{12}, x_t)$

row 2: $\big(0a_{11} + 1a_{21} + y_t \cdot 0,\;\; 0a_{12} + 1a_{22} + y_t \cdot 0,\;\; 0 \cdot 0 + 1 \cdot 0 + y_t \cdot 1\big) = (a_{21}, a_{22}, y_t)$

row 3: $\big(0a_{11} + 0a_{21} + 1 \cdot 0,\;\; 0a_{12} + 0a_{22} + 1 \cdot 0,\;\; 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1\big) = (0, 0, 1)$

Shift second: the two parts stay apart

$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(1a_{11} + 0a_{21} + x_t \cdot 0,\ \ 1a_{12} + 0a_{22} + x_t \cdot 0,\ \ 1 \cdot 0 + 0 \cdot 0 + x_t \cdot 1\big) = (a_{11}, a_{12}, x_t)$

row 2: $\big(0a_{11} + 1a_{21} + y_t \cdot 0,\ \ 0a_{12} + 1a_{22} + y_t \cdot 0,\ \ 0 \cdot 0 + 1 \cdot 0 + y_t \cdot 1\big) = (a_{21}, a_{22}, y_t)$

row 3: $\big(0a_{11} + 0a_{21} + 1 \cdot 0,\ \ 0a_{12} + 0a_{22} + 1 \cdot 0,\ \ 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1\big) = (0, 0, 1)$

$$= \begin{bmatrix} a_{11} & a_{12} & x_t \\ a_{21} & a_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

- Upper left: scale/rotate part

Shift second: the two parts stay apart

$$\begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

row 1: $\big(1a_{11} + 0a_{21} + x_t \cdot 0,\ \ 1a_{12} + 0a_{22} + x_t \cdot 0,\ \ 1 \cdot 0 + 0 \cdot 0 + x_t \cdot 1\big) = (a_{11}, a_{12}, x_t)$

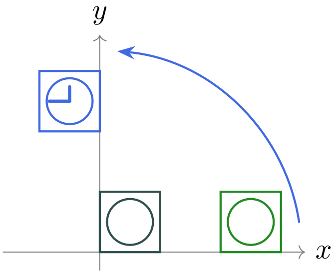
row 2: $\big(0a_{11} + 1a_{21} + y_t \cdot 0,\ \ 0a_{12} + 1a_{22} + y_t \cdot 0,\ \ 0 \cdot 0 + 1 \cdot 0 + y_t \cdot 1\big) = (a_{21}, a_{22}, y_t)$

row 3: $\big(0a_{11} + 0a_{21} + 1 \cdot 0,\ \ 0a_{12} + 0a_{22} + 1 \cdot 0,\ \ 0 \cdot 0 + 0 \cdot 0 + 1 \cdot 1\big) = (0, 0, 1)$

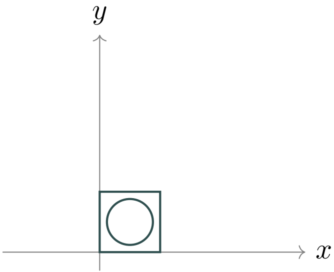
$$= \begin{bmatrix} a_{11} & a_{12} & x_t \\ a_{21} & a_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

- Upper left: scale/rotate part
- Last column: translation part

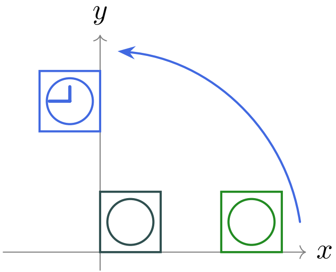
Same $\mathbf{M}$: turn first, then lift by 2



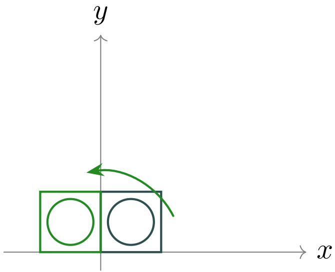
translate$(2, 0)$, then rotate(90°)



Same $\mathbf{M}$: turn first, then lift by 2

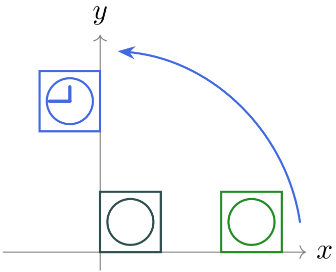


translate$(2, 0)$, then rotate(90°)

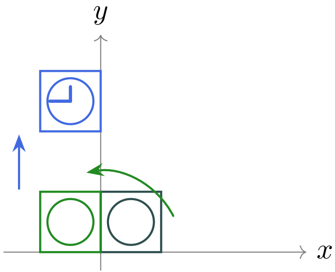


rotate(90°),

Same $\mathbf{M}$: turn first, then lift by 2



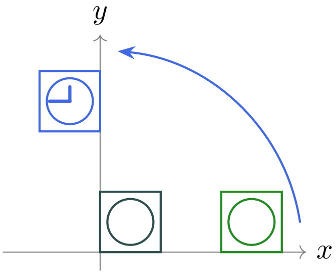
translate$(2, 0)$, then rotate$(90°)$



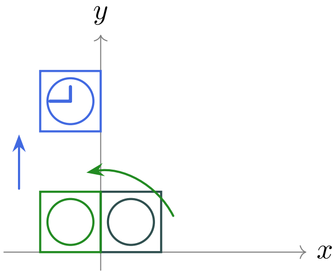
rotate$(90°)$, then translate$(0, 2)$

- Two recipes, same final clock

Same $\mathbf{M}$: turn first, then lift by 2



translate$(2, 0)$, then rotate$(90°)$

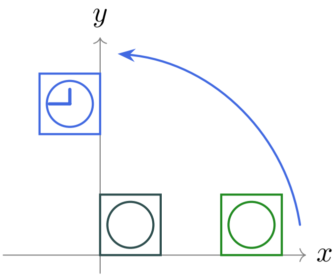


rotate$(90°)$, then translate$(0, 2)$

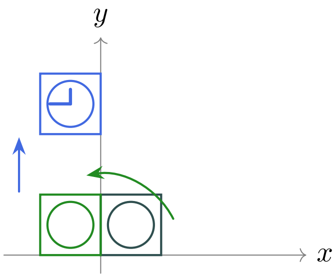
$$\mathbf{M} = \mathrm{translate}(0,2)\,\mathrm{rotate}(90^\circ) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

- Two recipes, same final clock

Same $\mathbf{M}$: turn first, then lift by 2



translate$(2, 0)$, then rotate(90°)

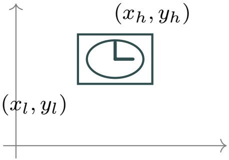


rotate(90°), then translate$(0, 2)$

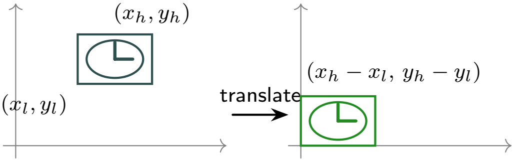
$$\mathbf{M} = \mathrm{translate}(0,2)\,\mathrm{rotate}(90^\circ) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

- Two recipes, same final clock
- Only turns and slides: rigid-body

Windowing: move, scale, move

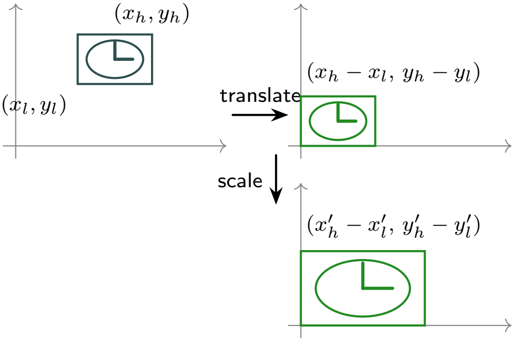


Windowing: move, scale, move



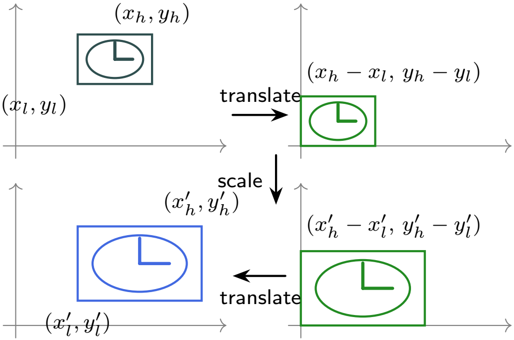
- 1. Move low corner to origin

Windowing: move, scale, move



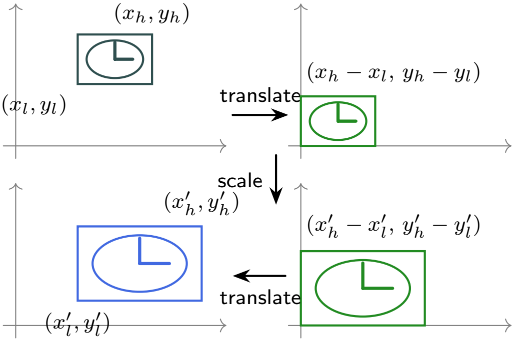
- 1. Move low corner to origin
- 2. Scale to the target size

Windowing: move, scale, move



- 1. Move low corner to origin
- 2. Scale to the target size
- 3. Move origin to target corner

Windowing: move, scale, move



- 1. Move low corner to origin
- 2. Scale to the target size
- 3. Move origin to target corner
- Write right to left

$$\text{window} = \text{translate}(x'_l, y'_l)\ \text{scale}\left(\frac{x'_h - x'_l}{x_h - x_l}, \frac{y'_h - y'_l}{y_h - y_l}\right) \text{translate}(-x_l, -y_l)$$

Multiply the three window matrices

$$s_x = \frac{x'_h - x'_l}{x_h - x_l}, \qquad s_y = \frac{y'_h - y'_l}{y_h - y_l} \qquad \text{(shorthand)}$$

Multiply the three window matrices

$$s_x = \frac{x'_h - x'_l}{x_h - x_l}, \qquad s_y = \frac{y'_h - y'_l}{y_h - y_l} \qquad \text{(shorthand)}$$

right pair first:

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_l \\ 0 & 1 & -y_l \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & -s_x x_l \\ 0 & s_y & -s_y y_l \\ 0 & 0 & 1 \end{bmatrix} \qquad s_x(-x_l) + 0(-y_l) + 0 \cdot 1$$

Multiply the three window matrices

$$s_x = \frac{x'_h - x'_l}{x_h - x_l}, \qquad s_y = \frac{y'_h - y'_l}{y_h - y_l} \qquad \text{(shorthand)}$$

right pair first:

$$\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_l \\ 0 & 1 & -y_l \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & -s_x x_l \\ 0 & s_y & -s_y y_l \\ 0 & 0 & 1 \end{bmatrix} \qquad s_x(-x_l) + 0(-y_l) + 0 \cdot 1$$

then the left translate:

$$\begin{bmatrix} 1 & 0 & x'_l \\ 0 & 1 & y'_l \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & -s_x x_l \\ 0 & s_y & -s_y y_l \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & x'_l - s_x x_l \\ 0 & s_y & y'_l - s_y y_l \\ 0 & 0 & 1 \end{bmatrix} \qquad 1(-s_x x_l) + 0 + x'_l \cdot 1$$

Clean up the corner: Equation 7.6

$$x'_l - s_x x_l = x'_l - \frac{(x'_h - x'_l)\, x_l}{x_h - x_l}$$

put in s_x

Clean up the corner: Equation 7.6

$$\begin{aligned} x'_l - s_x x_l &= x'_l - \frac{(x'_h - x'_l)\, x_l}{x_h - x_l} && \text{put in } s_x \\ &= \frac{x'_l(x_h - x_l) - (x'_h - x'_l)\, x_l}{x_h - x_l} && \text{common denominator} \end{aligned}$$

Clean up the corner: Equation 7.6

$$\begin{aligned}
x'_l - s_x x_l &= x'_l - \frac{(x'_h - x'_l)\, x_l}{x_h - x_l} && \text{put in } s_x \\
&= \frac{x'_l(x_h - x_l) - (x'_h - x'_l)\, x_l}{x_h - x_l} && \text{common denominator} \\
&= \frac{x'_l x_h - x'_l x_l - x'_h x_l + x'_l x_l}{x_h - x_l} && \text{expand}
\end{aligned}$$

Clean up the corner: Equation 7.6

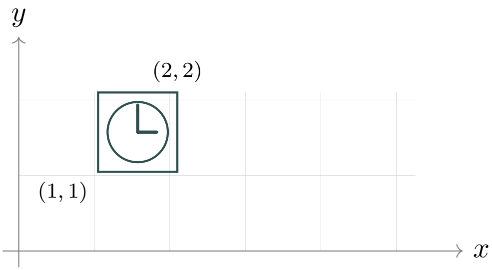
$$
\begin{aligned}
x'_l - s_x x_l &= x'_l - \frac{(x'_h - x'_l)\, x_l}{x_h - x_l} && \text{put in } s_x \\
&= \frac{x'_l(x_h - x_l) - (x'_h - x'_l)\, x_l}{x_h - x_l} && \text{common denominator} \\
&= \frac{x'_l x_h - x'_l x_l - x'_h x_l + x'_l x_l}{x_h - x_l} && \text{expand} \\
&= \frac{x'_l x_h - x'_h x_l}{x_h - x_l} && \text{cancel } x'_l x_l
\end{aligned}
$$

Clean up the corner: Equation 7.6

$$
\begin{aligned}
x'_l - s_x x_l &= x'_l - \frac{(x'_h - x'_l)\, x_l}{x_h - x_l} && \text{put in } s_x \\
&= \frac{x'_l(x_h - x_l) - (x'_h - x'_l)\, x_l}{x_h - x_l} && \text{common denominator} \\
&= \frac{x'_l x_h - x'_l x_l - x'_h x_l + x'_l x_l}{x_h - x_l} && \text{expand} \\
&= \frac{x'_l x_h - x'_h x_l}{x_h - x_l} && \text{cancel } x'_l x_l
\end{aligned}
$$

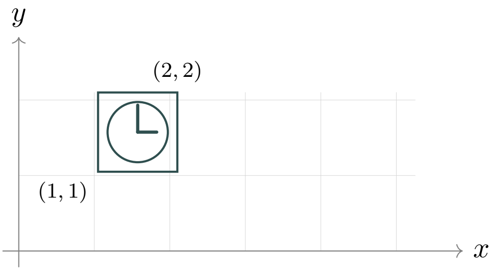
$$
\text{window} = \begin{bmatrix} \frac{x'_h - x'_l}{x_h - x_l} & 0 & \frac{x'_l x_h - x'_h x_l}{x_h - x_l} \\ 0 & \frac{y'_h - y'_l}{y_h - y_l} & \frac{y'_l y_h - y'_h y_l}{y_h - y_l} \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{same steps for } y \tag{7.6}
$$

Worked window: stretch and move the clock



$$[1,2] \times [1,2] \ \rightarrow \ [3,5] \times [0,1]$$

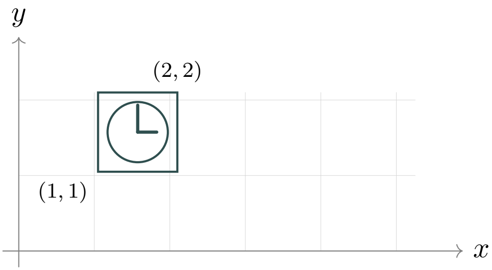
Worked window: stretch and move the clock



$$[1, 2] \times [1, 2] \;\rightarrow\; [3, 5] \times [0, 1]$$

$$\frac{x'_h - x'_l}{x_h - x_l} = \frac{5 - 3}{2 - 1} = \frac{2}{1} = 2$$

Worked window: stretch and move the clock

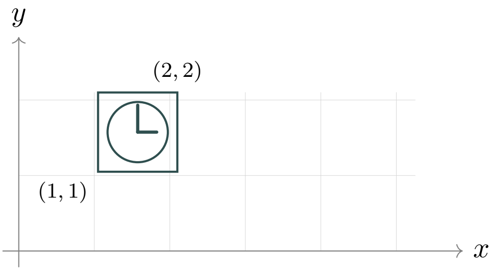


$$[1,2] \times [1,2] \;\rightarrow\; [3,5] \times [0,1]$$

$$\frac{x'_h - x'_l}{x_h - x_l} = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

$$\frac{y'_h - y'_l}{y_h - y_l} = \frac{1-0}{2-1} = \frac{1}{1} = 1$$

Worked window: stretch and move the clock



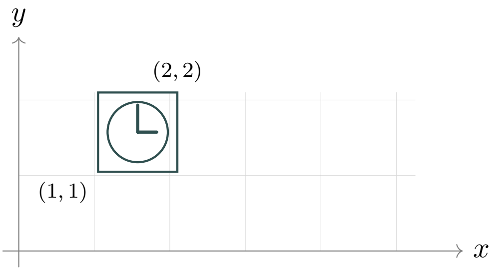
$$[1,2] \times [1,2] \;\rightarrow\; [3,5] \times [0,1]$$

$$\frac{x'_h - x'_l}{x_h - x_l} = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

$$\frac{y'_h - y'_l}{y_h - y_l} = \frac{1-0}{2-1} = \frac{1}{1} = 1$$

$$\frac{x'_l x_h - x'_h x_l}{x_h - x_l} = \frac{3 \cdot 2 - 5 \cdot 1}{2-1} = \frac{6-5}{1} = 1$$

Worked window: stretch and move the clock



$$[1,2]\times[1,2] \;\rightarrow\; [3,5]\times[0,1]$$

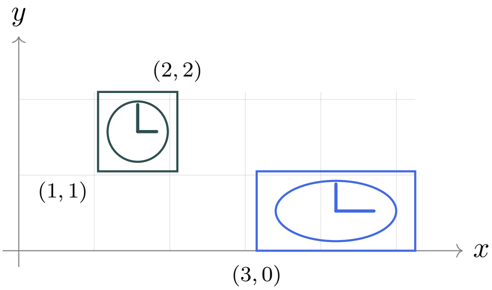
$$\frac{x'_h - x'_l}{x_h - x_l} = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

$$\frac{y'_h - y'_l}{y_h - y_l} = \frac{1-0}{2-1} = \frac{1}{1} = 1$$

$$\frac{x'_l x_h - x'_h x_l}{x_h - x_l} = \frac{3\cdot 2 - 5\cdot 1}{2-1} = \frac{6-5}{1} = 1$$

$$\frac{y'_l y_h - y'_h y_l}{y_h - y_l} = \frac{0\cdot 2 - 1\cdot 1}{2-1} = \frac{0-1}{1} = -1$$

Worked window: stretch and move the clock



$$[1, 2] \times [1, 2] \;\rightarrow\; [3, 5] \times [0, 1]$$

$$\frac{x'_h - x'_l}{x_h - x_l} = \frac{5 - 3}{2 - 1} = \frac{2}{1} = 2$$

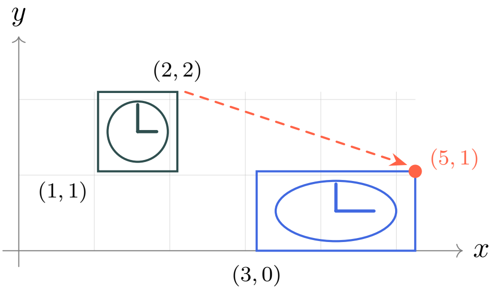
$$\frac{y'_h - y'_l}{y_h - y_l} = \frac{1 - 0}{2 - 1} = \frac{1}{1} = 1$$

$$\frac{x'_l x_h - x'_h x_l}{x_h - x_l} = \frac{3 \cdot 2 - 5 \cdot 1}{2 - 1} = \frac{6 - 5}{1} = 1$$

$$\frac{y'_l y_h - y'_h y_l}{y_h - y_l} = \frac{0 \cdot 2 - 1 \cdot 1}{2 - 1} = \frac{0 - 1}{1} = -1$$

$$\text{window} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Worked window: stretch and move the clock



$$[1,2]\times[1,2] \;\rightarrow\; [3,5]\times[0,1]$$

$$\begin{bmatrix} 2\cdot 2+0\cdot 2+1\cdot 1 \\ 0\cdot 2+1\cdot 2+(-1)\cdot 1 \\ 0\cdot 2+0\cdot 2+1\cdot 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

$$\frac{x'_h - x'_l}{x_h - x_l} = \frac{5-3}{2-1} = \frac{2}{1} = 2$$

$$\frac{y'_h - y'_l}{y_h - y_l} = \frac{1-0}{2-1} = \frac{1}{1} = 1$$

$$\frac{x'_l x_h - x'_h x_l}{x_h - x_l} = \frac{3\cdot 2 - 5\cdot 1}{2-1} = \frac{6-5}{1} = 1$$

$$\frac{y'_l y_h - y'_h y_l}{y_h - y_l} = \frac{0\cdot 2 - 1\cdot 1}{2-1} = \frac{0-1}{1} = -1$$

$$\text{window} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

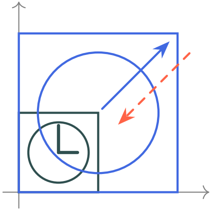
Recap: homogeneous coordinates

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Undoing Transformations

Every move has a way back

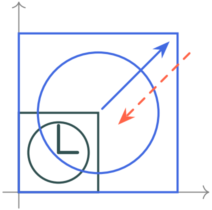
Undo a move with its opposite move



$$\text{scale}(s_x, s_y)^{-1}$$
$$= \text{scale}(1/s_x,\ 1/s_y)$$

Divide by each factor

Undo a move with its opposite move



$$\text{scale}(s_x, s_y)^{-1}$$
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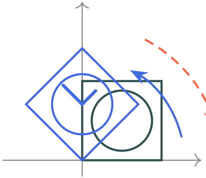
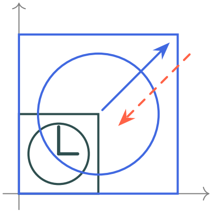
Divide by each factor



$$\text{rotate}(\phi)^{-1}$$
$$= \text{rotate}(-\phi)$$

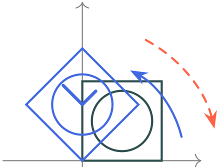
Flip the angle's sign

Undo a move with its opposite move



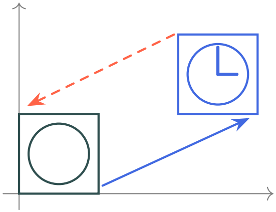
$$\mathrm{scale}(s_x, s_y)^{-1} = \mathrm{scale}(1/s_x,\ 1/s_y)$$

Divide by each factor



$$\mathrm{rotate}(\phi)^{-1} = \mathrm{rotate}(-\phi)$$

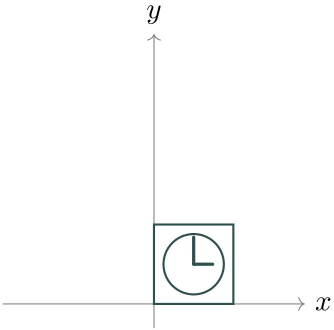
Flip the angle's sign



$$\mathrm{translate}(x_t, y_t)^{-1} = \mathrm{translate}(-x_t, -y_t)$$

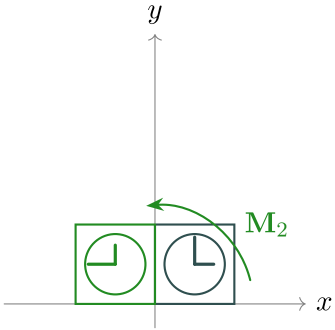
Slide back the other way

Undo a chain in reverse order



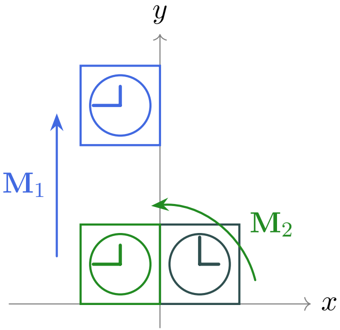
$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2 = \text{translate}(0, 2)\,\text{rotate}(90^\circ)$$

Undo a chain in reverse order



$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2 = \text{translate}(0, 2)\,\text{rotate}(90^\circ)$$

Undo a chain in reverse order



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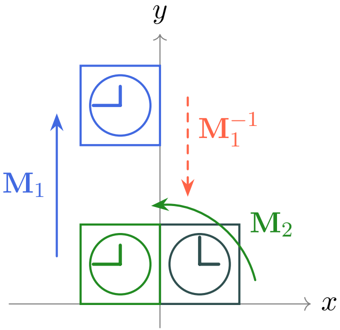
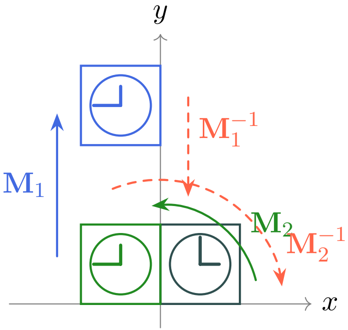
Undo a chain in reverse order



$$\mathbf{M} = \mathbf{M}_1 \mathbf{M}_2 = \text{translate}(0, 2)\, \text{rotate}(90^\circ)$$

- Undo the last move first

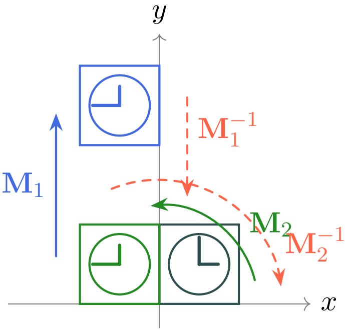
Undo a chain in reverse order



$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2 = \text{translate}(0,2)\,\text{rotate}(90^\circ)$$

- Undo the last move first

Undo a chain in reverse order

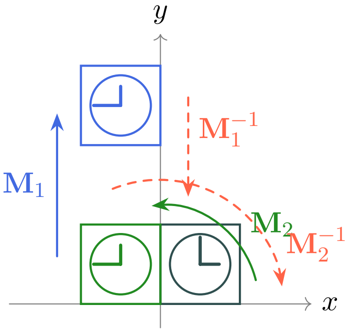


$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2 = \text{translate}(0,2)\,\text{rotate}(90^\circ)$$

- Undo the last move first

$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_n \;\Rightarrow\; \mathbf{M}^{-1} = \mathbf{M}_n^{-1}\cdots\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}$$

Undo a chain in reverse order



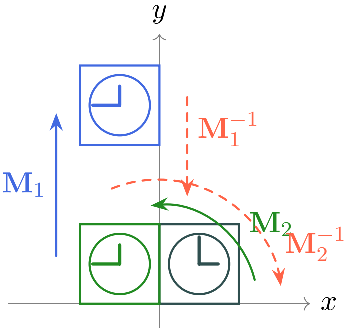
$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2 = \mathrm{translate}(0,2)\,\mathrm{rotate}(90^\circ)$$

- Undo the last move first

$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_n \;\Rightarrow\; \mathbf{M}^{-1} = \mathbf{M}_n^{-1}\cdots\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}$$

$$\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}\,\mathbf{M}_1\mathbf{M}_2$$

Undo a chain in reverse order



$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2 = \text{translate}(0,2)\,\text{rotate}(90^\circ)$$

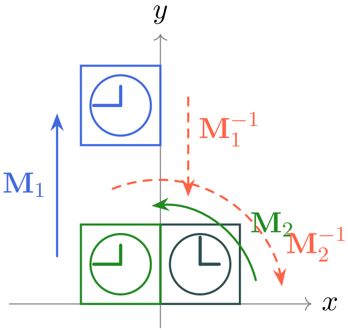
- Undo the last move first

$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_n \;\Rightarrow\; \mathbf{M}^{-1} = \mathbf{M}_n^{-1}\cdots\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}$$

$$\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}\,\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2^{-1}\,\mathbf{I}\,\mathbf{M}_2$$

inner pair cancels

Undo a chain in reverse order



$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2 = \text{translate}(0,2)\,\text{rotate}(90^\circ)$$

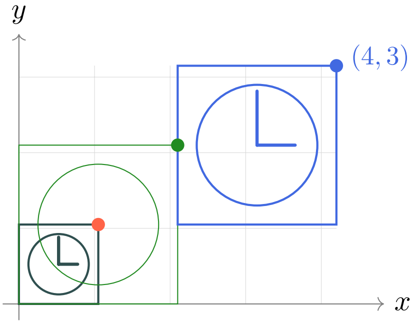
- Undo the last move first

$$\mathbf{M} = \mathbf{M}_1\mathbf{M}_2\cdots\mathbf{M}_n \;\Rightarrow\; \mathbf{M}^{-1} = \mathbf{M}_n^{-1}\cdots\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}$$

$$\mathbf{M}_2^{-1}\mathbf{M}_1^{-1}\,\mathbf{M}_1\mathbf{M}_2 = \mathbf{M}_2^{-1}\,\mathbf{I}\,\mathbf{M}_2 = \mathbf{M}_2^{-1}\mathbf{M}_2 = \mathbf{I}$$

inner pair cancels

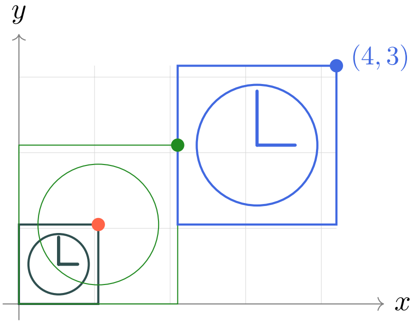
Quick check 2: undo this chain



$$\mathbf{M} = \text{translate}(2,1)\ \text{scale}(2,2)$$

Write $\mathbf{M}^{-1}$ as a product.

Quick check 2: undo this chain

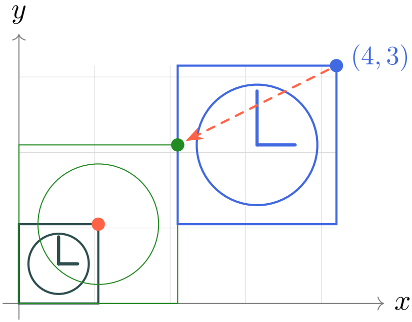


$$\mathbf{M} = \text{translate}(2, 1)\ \text{scale}(2, 2)$$

Write $\mathbf{M}^{-1}$ as a product.

$$\mathbf{M}^{-1} = \text{scale}(\tfrac{1}{2}, \tfrac{1}{2})\ \text{translate}(-2, -1)$$

Quick check 2: undo this chain



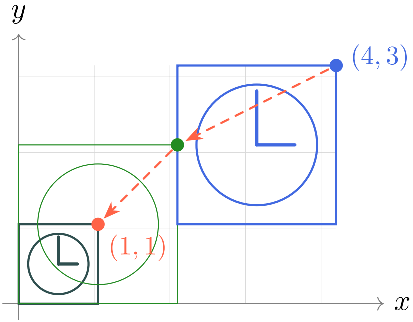
$$\mathbf{M} = \text{translate}(2,1)\ \text{scale}(2,2)$$

Write $\mathbf{M}^{-1}$ as a product.

$$\mathbf{M}^{-1} = \text{scale}(\tfrac{1}{2}, \tfrac{1}{2})\ \text{translate}(-2,-1)$$

$(4,3) \to (4-2,\ 3-1) = (2,2)$ slide back

Quick check 2: undo this chain



$$\mathbf{M} = \text{translate}(2, 1)\ \text{scale}(2, 2)$$

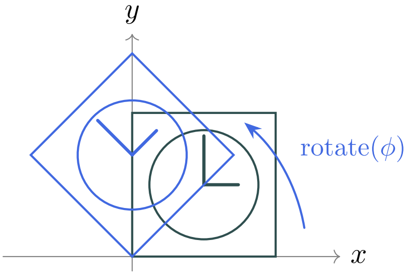
Write $\mathbf{M}^{-1}$ as a product.

$$\mathbf{M}^{-1} = \text{scale}(\tfrac{1}{2}, \tfrac{1}{2})\ \text{translate}(-2, -1)$$

$(4, 3) \to (4 - 2,\ 3 - 1) = (2, 2)$ slide back

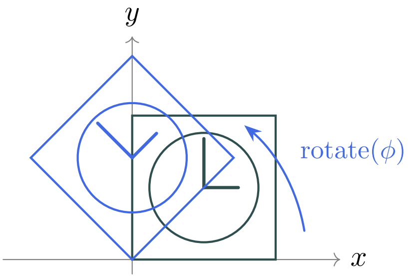
$(2, 2) \to (2 \cdot \frac{1}{2},\ 2 \cdot \frac{1}{2}) = (1, 1)$ halve

Rotation: the inverse is the transpose



$$\text{rotate}(\phi)^{\mathrm{T}} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{\mathrm{T}}$$

Rotation: the inverse is the transpose

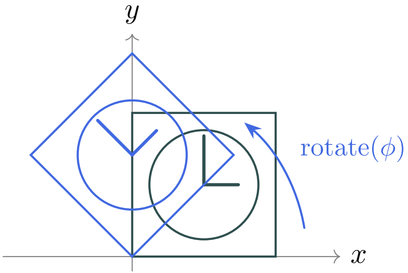


$$\mathrm{rotate}(\phi)^{\mathrm{T}} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{\mathrm{T}}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

rows become columns

Rotation: the inverse is the transpose

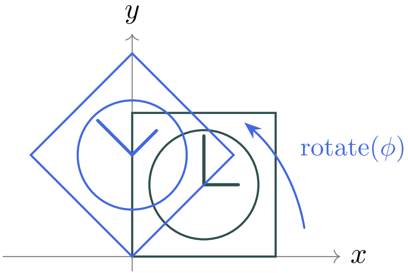


$$\text{rotate}(\phi)^{\mathrm{T}} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{\mathrm{T}}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$ rows become columns

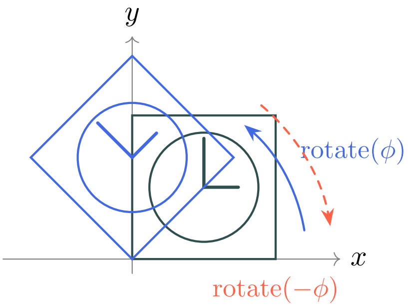
$$= \begin{bmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{bmatrix}$$ cos even, sin odd

Rotation: the inverse is the transpose



$$
\begin{aligned}
\mathrm{rotate}(\phi)^{\mathrm{T}} &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{\mathrm{T}} \\
&= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} && \text{rows become columns} \\
&= \begin{bmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{bmatrix} && \text{cos even, sin odd} \\
&= \mathrm{rotate}(-\phi) && \text{the rotation pattern}
\end{aligned}
$$

Rotation: the inverse is the transpose



$$
\begin{aligned}
\text{rotate}(\phi)^{\mathrm{T}} &= \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}^{\mathrm{T}} \\
&= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} && \text{rows become columns} \\
&= \begin{bmatrix} \cos(-\phi) & -\sin(-\phi) \\ \sin(-\phi) & \cos(-\phi) \end{bmatrix} && \text{cos even, sin odd} \\
&= \text{rotate}(-\phi) && \text{the rotation pattern}
\end{aligned}
$$

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I} \quad \Rightarrow \quad \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$$

Rigid-body inverse: slide back, then turn back

$$\mathbf{M} = \mathrm{translate}(x_t, y_t)\,\mathbf{R}, \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

turn, then slide

Rigid-body inverse: slide back, then turn back

$$\mathbf{M} = \text{translate}(x_t, y_t)\,\mathbf{R}, \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

turn, then slide

$$\mathbf{M}^{-1} = \mathbf{R}^{-1}\,\text{translate}(x_t, y_t)^{-1}$$

reverse order;

Rigid-body inverse: slide back, then turn back

$$\mathbf{M} = \text{translate}(x_t, y_t)\,\mathbf{R}, \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ turn, then slide

$$\mathbf{M}^{-1} = \mathbf{R}^{-1}\,\text{translate}(x_t, y_t)^{-1} = \mathbf{R}^{\mathrm{T}}\,\text{translate}(-x_t, -y_t)$$ reverse order; opposites

Rigid-body inverse: slide back, then turn back

$$\mathbf{M} = \text{translate}(x_t, y_t)\,\mathbf{R}, \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

turn, then slide

$$\mathbf{M}^{-1} = \mathbf{R}^{-1}\,\text{translate}(x_t, y_t)^{-1} = \mathbf{R}^{\mathrm{T}}\,\text{translate}(-x_t, -y_t)$$

reverse order; opposites

$$= \begin{bmatrix} r_{11} & r_{21} & 0 \\ r_{12} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_t \\ 0 & 1 & -y_t \\ 0 & 0 & 1 \end{bmatrix}$$

transpose swaps r_{12}, r_{21}

Rigid-body inverse: slide back, then turn back

$$\mathbf{M} = \text{translate}(x_t, y_t)\,\mathbf{R}, \quad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & 0 \\ r_{21} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

turn, then slide

$$\mathbf{M}^{-1} = \mathbf{R}^{-1}\,\text{translate}(x_t, y_t)^{-1} = \mathbf{R}^{\mathrm{T}}\,\text{translate}(-x_t, -y_t)$$

reverse order; opposites

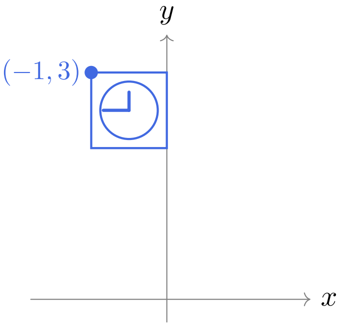
$$= \begin{bmatrix} r_{11} & r_{21} & 0 \\ r_{12} & r_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -x_t \\ 0 & 1 & -y_t \\ 0 & 0 & 1 \end{bmatrix}$$

transpose swaps r_{12}, r_{21}

$$= \begin{bmatrix} r_{11} & r_{21} & -(r_{11}x_t + r_{21}y_t) \\ r_{12} & r_{22} & -(r_{12}x_t + r_{22}y_t) \\ 0 & 0 & 1 \end{bmatrix}$$

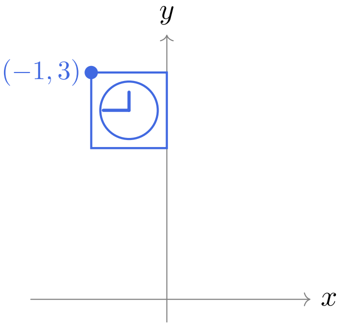
bottom row stays $0\;0\;1$

Worked: undo turn, then lift by 2



$$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} : \; r_{11} = 0, \; r_{12} = -1, \; r_{21} = 1, \; r_{22} = 0, \; (x_t, y_t) = (0, 2)$$

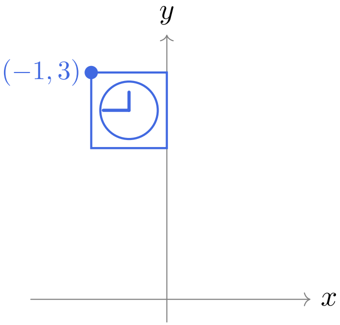
Worked: undo turn, then lift by 2



$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$: $r_{11} = 0,\ r_{12} = -1,\ r_{21} = 1,\ r_{22} = 0,\ (x_t, y_t) = (0, 2)$

$-(r_{11}x_t + r_{21}y_t) = -(0 \cdot 0 + 1 \cdot 2) = -2$

Worked: undo turn, then lift by 2

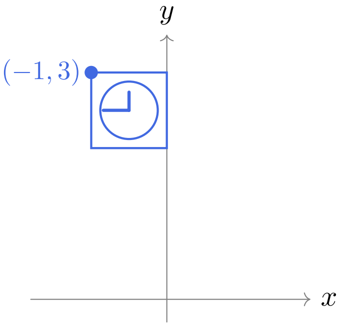


$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$: $r_{11} = 0,\ r_{12} = -1,\ r_{21} = 1,\ r_{22} = 0,\ (x_t, y_t) = (0, 2)$

$$-(r_{11}x_t + r_{21}y_t) = -(0 \cdot 0 + 1 \cdot 2) = -2$$

$$-(r_{12}x_t + r_{22}y_t) = -\big((-1) \cdot 0 + 0 \cdot 2\big) = 0$$

Worked: undo turn, then lift by 2



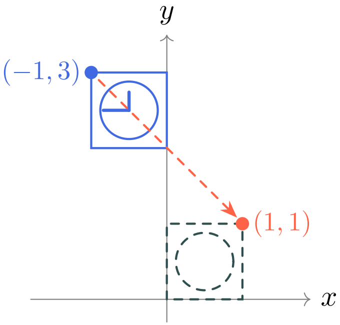
$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$: $r_{11} = 0,\ r_{12} = -1,\ r_{21} = 1,\ r_{22} = 0,\ (x_t, y_t) = (0, 2)$

$$-(r_{11}x_t + r_{21}y_t) = -(0 \cdot 0 + 1 \cdot 2) = -2$$

$$-(r_{12}x_t + r_{22}y_t) = -\big((-1) \cdot 0 + 0 \cdot 2\big) = 0$$

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Worked: undo turn, then lift by 2



$\mathbf{M} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$: $r_{11} = 0,\ r_{12} = -1,\ r_{21} = 1,\ r_{22} = 0,\ (x_t, y_t) = (0, 2)$

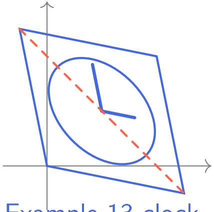
$-(r_{11}x_t + r_{21}y_t) = -(0 \cdot 0 + 1 \cdot 2) = -2$

$-(r_{12}x_t + r_{22}y_t) = -((-1) \cdot 0 + 0 \cdot 2) = 0$

$$\mathbf{M}^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{M}^{-1} \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot (-1) + 1 \cdot 3 + (-2) \cdot 1 \\ (-1) \cdot (-1) + 0 \cdot 3 + 0 \cdot 1 \\ 0 \cdot (-1) + 0 \cdot 3 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Rotate, scale, rotate: undo each piece



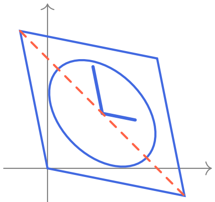
Example 13 clock

book: $\mathbf{M} = \mathbf{R}_1 \, \mathrm{scale}(\sigma_1, \sigma_2, \sigma_3) \, \mathbf{R}_2$

$$\mathbf{M}^{-1} = \mathbf{R}_2^{\mathrm{T}} \, \mathrm{scale}(1/\sigma_1, 1/\sigma_2, 1/\sigma_3) \, \mathbf{R}_1^{\mathrm{T}}$$

$$\mathbf{M} = \mathbf{R} \, \mathrm{scale}(1.5, 1) \, \mathbf{R}^{\mathrm{T}}$$

Rotate, scale, rotate: undo each piece



Example 13 clock

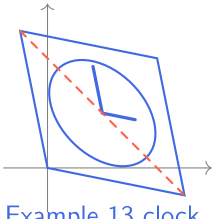
book: $\mathbf{M} = \mathbf{R}_1 \,\text{scale}(\sigma_1, \sigma_2, \sigma_3)\, \mathbf{R}_2$

$$\mathbf{M}^{-1} = \mathbf{R}_2^{\mathrm{T}} \,\text{scale}(1/\sigma_1, 1/\sigma_2, 1/\sigma_3)\, \mathbf{R}_1^{\mathrm{T}}$$

$$\mathbf{M} = \mathbf{R} \,\text{scale}(1.5, 1)\, \mathbf{R}^{\mathrm{T}}$$

$$\mathbf{M}^{-1} = (\mathbf{R}^{\mathrm{T}})^{-1} \,\text{scale}(1.5, 1)^{-1}\, \mathbf{R}^{-1}$$ reverse order

Rotate, scale, rotate: undo each piece



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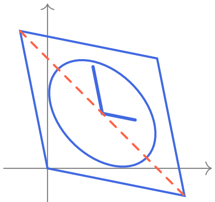
$$= (\mathbf{R}^{\mathrm{T}})^{\mathrm{T}} \, \mathrm{scale}(1/1.5,\ 1) \, \mathbf{R}^{\mathrm{T}} \qquad \text{transpose, divide}$$

Rotate, scale, rotate: undo each piece



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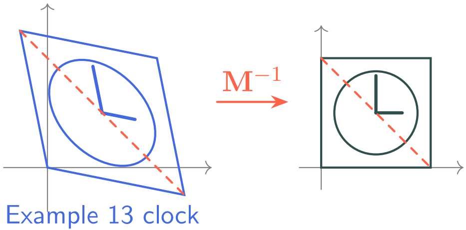
$$= (\mathbf{R}^{\mathrm{T}})^{\mathrm{T}} \,\mathrm{scale}(1/1.5,\ 1)\, \mathbf{R}^{\mathrm{T}} \qquad \text{transpose, divide}$$

$$= \mathbf{R} \,\mathrm{scale}(2/3,\ 1)\, \mathbf{R}^{\mathrm{T}} \qquad (\mathbf{R}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{R}$$

Rotate, scale, rotate: undo each piece



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- Squash back along the diagonal

No geometry? Use cofactors

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} & {\color{red}a_{12}} & \end{array} \\ \begin{bmatrix} 1 & \boxed{1} & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix} \end{array}$$

- Every entry has a cofactor

No geometry? Use cofactors

$$\mathbf{A} = \begin{array}{c} \color{red}{a_{12}} \\ \begin{bmatrix} \cancel{1} & \boxed{\cancel{1}} & \cancel{2} \\ \boxed{1} & \cancel{3} & \boxed{4} \\ \boxed{0} & \cancel{2} & \boxed{5} \end{bmatrix} \end{array}$$

- Every entry has a cofactor
- Cross out its row and column

No geometry? Use cofactors

$$\mathbf{A} = \begin{array}{c} \begin{array}{ccc} & {\color{red}a_{12}} & \end{array} \\ \begin{bmatrix} 1 & \boxed{1} & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix} \end{array}$$

signs

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

- Every entry has a cofactor
- Cross out its row and column
- Sign from the checkerboard

No geometry? Use cofactors

$$\begin{array}{cc} & \begin{array}{ccc} & a_{12} & \end{array} \\ \mathbf{A} = & \begin{bmatrix} 1 & \boxed{1} & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix} \end{array}$$

signs

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

$$a_{12}^c = -\begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} = -(1 \cdot 5 - 4 \cdot 0) = -5$$

- Every entry has a cofactor
- Cross out its row and column
- Sign from the checkerboard

$$\begin{vmatrix} a & A \\ b & B \end{vmatrix} = aB - Ab$$

Expand along row 1 to get $|\mathbf{A}|$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

$$|\mathbf{A}| = a_{11}a_{11}^c + a_{12}a_{12}^c + a_{13}a_{13}^c$$ entries times cofactors

Expand along row 1 to get $|\mathbf{A}|$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\begin{aligned} |\mathbf{A}| &= a_{11}a_{11}^c + a_{12}a_{12}^c + a_{13}a_{13}^c \\ &= 1\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} \end{aligned}$$

entries times cofactors

cross out, sign

Expand along row 1 to get $|\mathbf{A}|$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\begin{aligned} |\mathbf{A}| &= a_{11}a^c_{11} + a_{12}a^c_{12} + a_{13}a^c_{13} && \text{entries times cofactors} \\ &= 1\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} && \text{cross out, sign} \\ &= 1(3\cdot 5 - 4\cdot 2) - 1(1\cdot 5 - 4\cdot 0) + 2(1\cdot 2 - 3\cdot 0) && aB - Ab \end{aligned}$$

Expand along row 1 to get $|\mathbf{A}|$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$$

$$\begin{aligned}
|\mathbf{A}| &= a_{11}a_{11}^c + a_{12}a_{12}^c + a_{13}a_{13}^c && \text{entries times cofactors} \\
&= 1\begin{vmatrix} 3 & 4 \\ 2 & 5 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 0 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} && \text{cross out, sign} \\
&= 1(3 \cdot 5 - 4 \cdot 2) - 1(1 \cdot 5 - 4 \cdot 0) + 2(1 \cdot 2 - 3 \cdot 0) && aB - Ab \\
&= 1(15 - 8) - 1(5 - 0) + 2(2 - 0) && \text{multiply}
\end{aligned}$$

Expand along row 1 to get $|\mathbf{A}|$

$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}$

$$
\begin{aligned}
|\mathbf{A}| &= a_{11}a_{11}^c + a_{12}a_{12}^c + a_{13}a_{13}^c && \text{entries times cofactors} \\
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&= 1(3 \cdot 5 - 4 \cdot 2) - 1(1 \cdot 5 - 4 \cdot 0) + 2(1 \cdot 2 - 3 \cdot 0) && aB - Ab \\
&= 1(15 - 8) - 1(5 - 0) + 2(2 - 0) && \text{multiply} \\
&= 7 - 5 + 4 && \text{subtract}
\end{aligned}
$$

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&= 1(3 \cdot 5 - 4 \cdot 2) - 1(1 \cdot 5 - 4 \cdot 0) + 2(1 \cdot 2 - 3 \cdot 0) && aB - Ab \\
&= 1(15 - 8) - 1(5 - 0) + 2(2 - 0) && \text{multiply} \\
&= 7 - 5 + 4 && \text{subtract} \\
&= 6 && \text{add}
\end{aligned}$$

Inverse: cofactors, transposed, over $|\mathbf{A}|$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} a^c_{11} & a^c_{21} & a^c_{31} \\ a^c_{12} & a^c_{22} & a^c_{32} \\ a^c_{13} & a^c_{23} & a^c_{33} \end{bmatrix}$$

adjoint: cofactors, transposed

Inverse: cofactors, transposed, over $|\mathbf{A}|$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} a^c_{11} & a^c_{21} & a^c_{31} \\ a^c_{12} & a^c_{22} & a^c_{32} \\ a^c_{13} & a^c_{23} & a^c_{33} \end{bmatrix}$$ adjoint: cofactors, transposed

row 1 $\cdot$ column 1: $a_{11}a^c_{11} + a_{12}a^c_{12} + a_{13}a^c_{13} = |\mathbf{A}|$ own row's cofactors

Inverse: cofactors, transposed, over $|\mathbf{A}|$

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row 1 · column 1: $a_{11}a^c_{11} + a_{12}a^c_{12} + a_{13}a^c_{13} = |\mathbf{A}|$ own row's cofactors

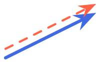
row 2 · column 1: $a_{21}a^c_{11} + a_{22}a^c_{12} + a_{23}a^c_{13} = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$

Inverse: cofactors, transposed, over $|\mathbf{A}|$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \begin{bmatrix} a_{11}^c & a_{21}^c & a_{31}^c \\ a_{12}^c & a_{22}^c & a_{32}^c \\ a_{13}^c & a_{23}^c & a_{33}^c \end{bmatrix}$$ adjoint: cofactors, transposed

row 1 · column 1: $a_{11}a_{11}^c + a_{12}a_{12}^c + a_{13}a_{13}^c = |\mathbf{A}|$ own row's cofactors

row 2 · column 1: $a_{21}a_{11}^c + a_{22}a_{12}^c + a_{23}a_{13}^c = \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$



- Two equal rows: zero volume

Inverse: cofactors, transposed, over $|\mathbf{A}|$

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\begin{bmatrix} a^c_{11} & a^c_{21} & a^c_{31} \\ a^c_{12} & a^c_{22} & a^c_{32} \\ a^c_{13} & a^c_{23} & a^c_{33} \end{bmatrix}$$ adjoint: cofactors, transposed

row 1 · column 1: $a_{11}a^c_{11} + a_{12}a^c_{12} + a_{13}a^c_{13} = |\mathbf{A}|$ own row's cofactors

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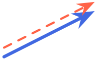

area 0

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = |\mathbf{A}|\,\mathbf{I} \;\Rightarrow\; \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\,\mathrm{adj}(\mathbf{A})$$

- Two equal rows: zero volume

Example 6: nine cofactors, then divide by 6

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}, \quad |\mathbf{A}| = 6$$

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$$\begin{bmatrix} \left| \begin{smallmatrix} 3 & 4 \\ 2 & 5 \end{smallmatrix} \right| = 15 - 8 = 7 & & \\ & & \\ & & \end{bmatrix}$$

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$$\begin{bmatrix} \left|\begin{smallmatrix} 3 & 4 \\ 2 & 5 \end{smallmatrix}\right| = 15 - 8 = 7 & -\left|\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right| = -(5-4) = -1 \\ & \\ & \\ & \end{bmatrix}$$

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$$\begin{bmatrix} \left|\begin{smallmatrix} 3 & 4 \\ 2 & 5 \end{smallmatrix}\right| = 15 - 8 = 7 & -\left|\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right| = -(5-4) = -1 & \left|\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right| = 4 - 6 = -2 \\ -\left|\begin{smallmatrix} 1 & 4 \\ 0 & 5 \end{smallmatrix}\right| = -(5-0) = -5 & & \\ & & \end{bmatrix}$$

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$$\begin{bmatrix} \left|\begin{smallmatrix} 3 & 4 \\ 2 & 5 \end{smallmatrix}\right| = 15 - 8 = 7 & -\left|\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right| = -(5-4) = -1 & \left|\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right| = 4 - 6 = -2 \\ \\ -\left|\begin{smallmatrix} 1 & 4 \\ 0 & 5 \end{smallmatrix}\right| = -(5-0) = -5 & \left|\begin{smallmatrix} 1 & 2 \\ 0 & 5 \end{smallmatrix}\right| = 5 - 0 = 5 & \\ \\ & & \end{bmatrix}$$

Example 6: nine cofactors, then divide by 6

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}, \quad |\mathbf{A}| = 6$$

$$\begin{bmatrix} \left|\begin{smallmatrix} 3 & 4 \\ 2 & 5 \end{smallmatrix}\right| = 15 - 8 = 7 & -\left|\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right| = -(5-4) = -1 & \left|\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right| = 4 - 6 = -2 \\ -\left|\begin{smallmatrix} 1 & 4 \\ 0 & 5 \end{smallmatrix}\right| = -(5-0) = -5 & \left|\begin{smallmatrix} 1 & 2 \\ 0 & 5 \end{smallmatrix}\right| = 5 - 0 = 5 & -\left|\begin{smallmatrix} 1 & 2 \\ 1 & 4 \end{smallmatrix}\right| = -(4-2) = -2 \\ & & \end{bmatrix}$$

Example 6: nine cofactors, then divide by 6

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}, \quad |\mathbf{A}| = 6$$

$$\begin{bmatrix} \left|\begin{smallmatrix} 3 & 4 \\ 2 & 5 \end{smallmatrix}\right| = 15 - 8 = 7 & -\left|\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right| = -(5-4) = -1 & \left|\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right| = 4 - 6 = -2 \\ -\left|\begin{smallmatrix} 1 & 4 \\ 0 & 5 \end{smallmatrix}\right| = -(5-0) = -5 & \left|\begin{smallmatrix} 1 & 2 \\ 0 & 5 \end{smallmatrix}\right| = 5 - 0 = 5 & -\left|\begin{smallmatrix} 1 & 2 \\ 1 & 4 \end{smallmatrix}\right| = -(4-2) = -2 \\ \left|\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}\right| = 2 - 0 = 2 & & \end{bmatrix}$$

Example 6: nine cofactors, then divide by 6

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}, \quad |\mathbf{A}| = 6$$

$$\begin{bmatrix} \left|\begin{smallmatrix} 3 & 4 \\ 2 & 5 \end{smallmatrix}\right| = 15 - 8 = 7 & -\left|\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right| = -(5-4) = -1 & \left|\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right| = 4 - 6 = -2 \\ -\left|\begin{smallmatrix} 1 & 4 \\ 0 & 5 \end{smallmatrix}\right| = -(5-0) = -5 & \left|\begin{smallmatrix} 1 & 2 \\ 0 & 5 \end{smallmatrix}\right| = 5 - 0 = 5 & -\left|\begin{smallmatrix} 1 & 2 \\ 1 & 4 \end{smallmatrix}\right| = -(4-2) = -2 \\ \left|\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}\right| = 2 - 0 = 2 & -\left|\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right| = -(2-0) = -2 & \end{bmatrix}$$

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Example 6: nine cofactors, then divide by 6

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 0 & 2 & 5 \end{bmatrix}, \quad |\mathbf{A}| = 6 \qquad\qquad \mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} 7 & -1 & -2 \\ -5 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \left|\begin{smallmatrix} 3 & 4 \\ 2 & 5 \end{smallmatrix}\right| = 15 - 8 = 7 & -\left|\begin{smallmatrix} 1 & 2 \\ 2 & 5 \end{smallmatrix}\right| = -(5-4) = -1 & \left|\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right| = 4 - 6 = -2 \\ -\left|\begin{smallmatrix} 1 & 4 \\ 0 & 5 \end{smallmatrix}\right| = -(5-0) = -5 & \left|\begin{smallmatrix} 1 & 2 \\ 0 & 5 \end{smallmatrix}\right| = 5 - 0 = 5 & -\left|\begin{smallmatrix} 1 & 2 \\ 1 & 4 \end{smallmatrix}\right| = -(4-2) = -2 \\ \left|\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}\right| = 2 - 0 = 2 & -\left|\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right| = -(2-0) = -2 & \left|\begin{smallmatrix} 1 & 1 \\ 1 & 3 \end{smallmatrix}\right| = 3 - 1 = 2 \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ $\cdot$ column j of the adjoint:

$$\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ · column j of the adjoint:

$$\begin{bmatrix} \begin{aligned} &1 \cdot 7 + 1 \cdot (-5) + 2 \cdot 2 \\ &\quad = 7 - 5 + 4 = 6 \end{aligned} & & \\ & & \\ & & \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ $\cdot$ column j of the adjoint:

$$\begin{bmatrix} \begin{array}{c} 1\cdot 7 + 1\cdot(-5) + 2\cdot 2 \\ = 7 - 5 + 4 = 6 \end{array} & \begin{array}{c} 1\cdot(-1) + 1\cdot 5 + 2\cdot(-2) \\ = -1 + 5 - 4 = 0 \end{array} & \\ & & \\ & & \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ · column j of the adjoint:

$$\begin{bmatrix} \begin{array}{c} 1 \cdot 7 + 1 \cdot (-5) + 2 \cdot 2 \\ = 7 - 5 + 4 = 6 \end{array} & \begin{array}{c} 1 \cdot (-1) + 1 \cdot 5 + 2 \cdot (-2) \\ = -1 + 5 - 4 = 0 \end{array} & \begin{array}{c} 1 \cdot (-2) + 1 \cdot (-2) + 2 \cdot 2 \\ = -2 - 2 + 4 = 0 \end{array} \\ & & \\ & & \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ $\cdot$ column j of the adjoint:

$$\begin{bmatrix} \begin{array}{c} 1 \cdot 7 + 1 \cdot (-5) + 2 \cdot 2 \\ = 7 - 5 + 4 = 6 \end{array} & \begin{array}{c} 1 \cdot (-1) + 1 \cdot 5 + 2 \cdot (-2) \\ = -1 + 5 - 4 = 0 \end{array} & \begin{array}{c} 1 \cdot (-2) + 1 \cdot (-2) + 2 \cdot 2 \\ = -2 - 2 + 4 = 0 \end{array} \\ \\ \begin{array}{c} 1 \cdot 7 + 3 \cdot (-5) + 4 \cdot 2 \\ = 7 - 15 + 8 = 0 \end{array} & & \\ \\ & & \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ $\cdot$ column j of the adjoint:

$$\begin{bmatrix} \begin{array}{c} 1\cdot 7+1\cdot(-5)+2\cdot 2 \\ =7-5+4=6 \end{array} & \begin{array}{c} 1\cdot(-1)+1\cdot 5+2\cdot(-2) \\ =-1+5-4=0 \end{array} & \begin{array}{c} 1\cdot(-2)+1\cdot(-2)+2\cdot 2 \\ =-2-2+4=0 \end{array} \\ \\ \begin{array}{c} 1\cdot 7+3\cdot(-5)+4\cdot 2 \\ =7-15+8=0 \end{array} & \begin{array}{c} 1\cdot(-1)+3\cdot 5+4\cdot(-2) \\ =-1+15-8=6 \end{array} & \\ \\ & & \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ · column j of the adjoint:

$$\begin{bmatrix} \begin{matrix} 1 \cdot 7 + 1 \cdot (-5) + 2 \cdot 2 \\ = 7 - 5 + 4 = 6 \end{matrix} & \begin{matrix} 1 \cdot (-1) + 1 \cdot 5 + 2 \cdot (-2) \\ = -1 + 5 - 4 = 0 \end{matrix} & \begin{matrix} 1 \cdot (-2) + 1 \cdot (-2) + 2 \cdot 2 \\ = -2 - 2 + 4 = 0 \end{matrix} \\ \\ \begin{matrix} 1 \cdot 7 + 3 \cdot (-5) + 4 \cdot 2 \\ = 7 - 15 + 8 = 0 \end{matrix} & \begin{matrix} 1 \cdot (-1) + 3 \cdot 5 + 4 \cdot (-2) \\ = -1 + 15 - 8 = 6 \end{matrix} & \begin{matrix} 1 \cdot (-2) + 3 \cdot (-2) + 4 \cdot 2 \\ = -2 - 6 + 8 = 0 \end{matrix} \\ \\ & & \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ $\cdot$ column j of the adjoint:

$$\begin{bmatrix} \begin{array}{c} 1\cdot 7+1\cdot(-5)+2\cdot 2 \\ =7-5+4=6 \end{array} & \begin{array}{c} 1\cdot(-1)+1\cdot 5+2\cdot(-2) \\ =-1+5-4=0 \end{array} & \begin{array}{c} 1\cdot(-2)+1\cdot(-2)+2\cdot 2 \\ =-2-2+4=0 \end{array} \\ \\ \begin{array}{c} 1\cdot 7+3\cdot(-5)+4\cdot 2 \\ =7-15+8=0 \end{array} & \begin{array}{c} 1\cdot(-1)+3\cdot 5+4\cdot(-2) \\ =-1+15-8=6 \end{array} & \begin{array}{c} 1\cdot(-2)+3\cdot(-2)+4\cdot 2 \\ =-2-6+8=0 \end{array} \\ \\ \begin{array}{c} 0\cdot 7+2\cdot(-5)+5\cdot 2 \\ =0-10+10=0 \end{array} & & \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ $\cdot$ column j of the adjoint:

$$\begin{bmatrix} \begin{array}{c} 1 \cdot 7 + 1 \cdot (-5) + 2 \cdot 2 \\ = 7 - 5 + 4 = 6 \end{array} & \begin{array}{c} 1 \cdot (-1) + 1 \cdot 5 + 2 \cdot (-2) \\ = -1 + 5 - 4 = 0 \end{array} & \begin{array}{c} 1 \cdot (-2) + 1 \cdot (-2) + 2 \cdot 2 \\ = -2 - 2 + 4 = 0 \end{array} \\ \\ \begin{array}{c} 1 \cdot 7 + 3 \cdot (-5) + 4 \cdot 2 \\ = 7 - 15 + 8 = 0 \end{array} & \begin{array}{c} 1 \cdot (-1) + 3 \cdot 5 + 4 \cdot (-2) \\ = -1 + 15 - 8 = 6 \end{array} & \begin{array}{c} 1 \cdot (-2) + 3 \cdot (-2) + 4 \cdot 2 \\ = -2 - 6 + 8 = 0 \end{array} \\ \\ \begin{array}{c} 0 \cdot 7 + 2 \cdot (-5) + 5 \cdot 2 \\ = 0 - 10 + 10 = 0 \end{array} & \begin{array}{c} 0 \cdot (-1) + 2 \cdot 5 + 5 \cdot (-2) \\ = 0 + 10 - 10 = 0 \end{array} & \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ $\cdot$ column j of the adjoint:

$$\begin{bmatrix} \begin{matrix} 1\cdot 7 + 1\cdot(-5) + 2\cdot 2 \\ = 7 - 5 + 4 = 6 \end{matrix} & \begin{matrix} 1\cdot(-1) + 1\cdot 5 + 2\cdot(-2) \\ = -1 + 5 - 4 = 0 \end{matrix} & \begin{matrix} 1\cdot(-2) + 1\cdot(-2) + 2\cdot 2 \\ = -2 - 2 + 4 = 0 \end{matrix} \\ \\ \begin{matrix} 1\cdot 7 + 3\cdot(-5) + 4\cdot 2 \\ = 7 - 15 + 8 = 0 \end{matrix} & \begin{matrix} 1\cdot(-1) + 3\cdot 5 + 4\cdot(-2) \\ = -1 + 15 - 8 = 6 \end{matrix} & \begin{matrix} 1\cdot(-2) + 3\cdot(-2) + 4\cdot 2 \\ = -2 - 6 + 8 = 0 \end{matrix} \\ \\ \begin{matrix} 0\cdot 7 + 2\cdot(-5) + 5\cdot 2 \\ = 0 - 10 + 10 = 0 \end{matrix} & \begin{matrix} 0\cdot(-1) + 2\cdot 5 + 5\cdot(-2) \\ = 0 + 10 - 10 = 0 \end{matrix} & \begin{matrix} 0\cdot(-2) + 2\cdot(-2) + 5\cdot 2 \\ = 0 - 4 + 10 = 6 \end{matrix} \end{bmatrix}$$

Check: $\mathbf{A}$ times its inverse is $\mathbf{I}$

row i of $\mathbf{A}$ $\cdot$ column j of the adjoint:

$$\begin{bmatrix} \begin{gathered}1\cdot 7+1\cdot(-5)+2\cdot 2\\ =7-5+4=6\end{gathered} & \begin{gathered}1\cdot(-1)+1\cdot 5+2\cdot(-2)\\ =-1+5-4=0\end{gathered} & \begin{gathered}1\cdot(-2)+1\cdot(-2)+2\cdot 2\\ =-2-2+4=0\end{gathered} \\ \\ \begin{gathered}1\cdot 7+3\cdot(-5)+4\cdot 2\\ =7-15+8=0\end{gathered} & \begin{gathered}1\cdot(-1)+3\cdot 5+4\cdot(-2)\\ =-1+15-8=6\end{gathered} & \begin{gathered}1\cdot(-2)+3\cdot(-2)+4\cdot 2\\ =-2-6+8=0\end{gathered} \\ \\ \begin{gathered}0\cdot 7+2\cdot(-5)+5\cdot 2\\ =0-10+10=0\end{gathered} & \begin{gathered}0\cdot(-1)+2\cdot 5+5\cdot(-2)\\ =0+10-10=0\end{gathered} & \begin{gathered}0\cdot(-2)+2\cdot(-2)+5\cdot 2\\ =0-4+10=6\end{gathered} \end{bmatrix}$$

$$\mathbf{A}\mathbf{A}^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \mathbf{I}$$

- Own row: 6. Other rows: 0.

Recap: undoing transformations

$$\mathbf{M}^{-1} = \mathbf{M}_n^{-1} \cdots \mathbf{M}_2^{-1} \mathbf{M}_1^{-1}$$

Summary

$$\mathbf{M} = \mathbf{R}\mathbf{S}$$

Rightmost matrix acts first

$$\begin{bmatrix} m_{11} & m_{12} & x_t \\ m_{21} & m_{22} & y_t \\ 0 & 0 & 1 \end{bmatrix}$$

Linear part plus translation column

$$[x \;\; y \;\; 1]^{\mathrm{T}} \qquad [x \;\; y \;\; 0]^{\mathrm{T}}$$

1: a location. 0: a direction.

$$\mathrm{translate}(x_l', y_l')\,\mathrm{scale}(\ldots)\,\mathrm{translate}(-x_l, -y_l)$$

Window: move, scale, move

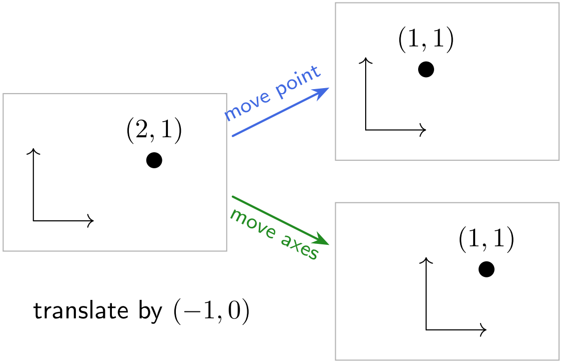
$$\mathbf{M}^{-1} = \mathbf{M}_n^{-1} \cdots \mathbf{M}_1^{-1}, \quad \mathbf{R}^{-1} = \mathbf{R}^{\mathrm{T}}$$

Undo in reverse order

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}\,\mathrm{adj}(\mathbf{A})$$

Cofactors, transposed, over the determinant

Next lecture: reference frames and the scene graph



translate by $(-1,0)$

- Move the point, or the axes?
- Frames and the scene graph

Shirley §2.4.5, §7.5, §12.2, §12.1–12.1.1