Lecture 7: 2×2 Matrices

Area, Matrix Rules, Scale, Shear, Rotation

CS 453 — Computer Graphics

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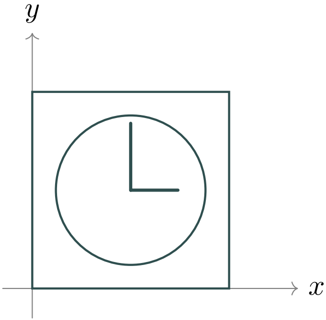
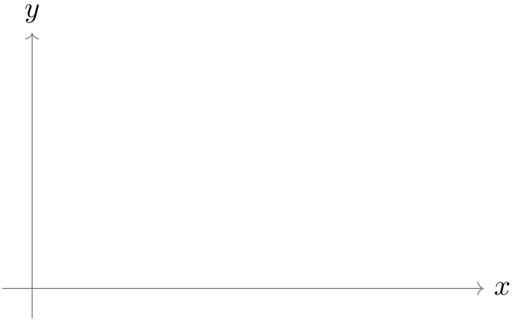
Forman Christian University

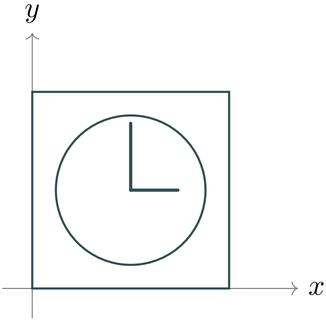
Matrices move every point of a shape



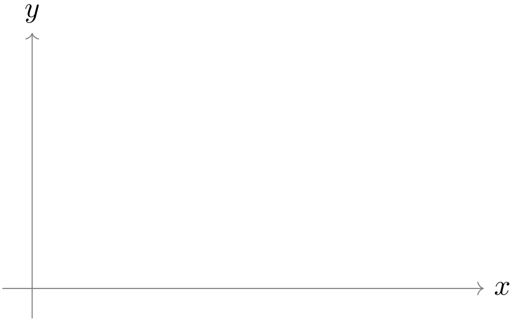


A clock made of points

Matrices move every point of a shape



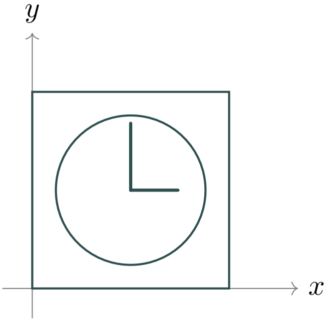
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$



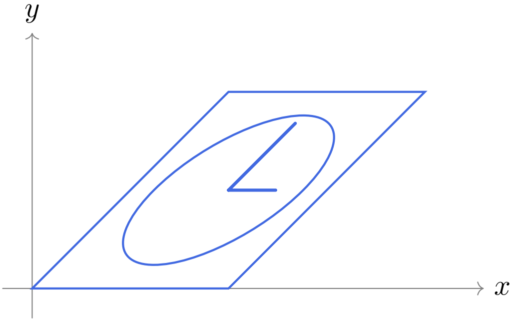
A clock made of points

One grid of four numbers

Matrices move every point of a shape



$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

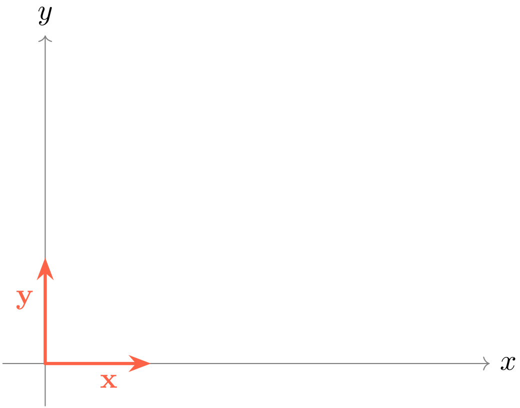


A clock made of points

One grid of four numbers

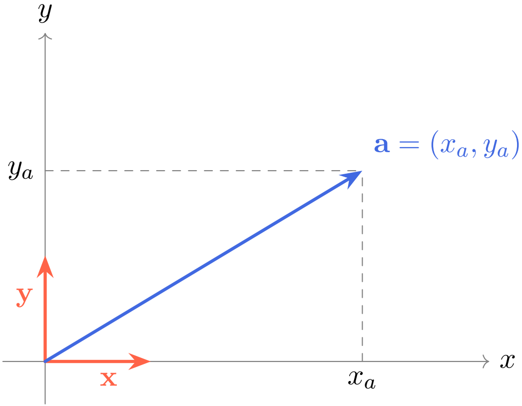
Every point moves at once

Our conventions in one picture



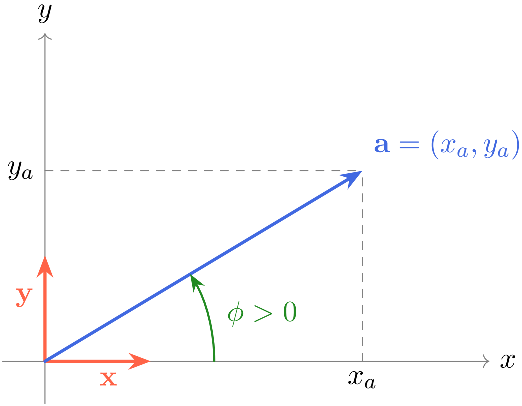
- x right, y up

Our conventions in one picture



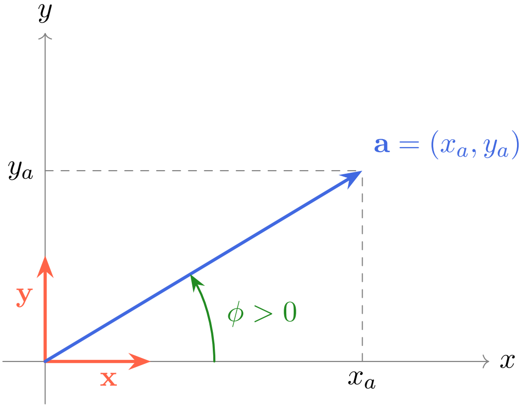
- x right, y up

Our conventions in one picture



- x right, y up
- Positive angles turn counterclockwise

Our conventions in one picture



$$\mathbf{M} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} x'_a \\ y'_a \end{bmatrix}$$

matrix

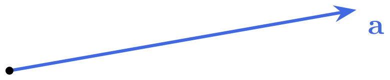
column vector

- x right, y up
- Positive angles turn counterclockwise
- Matrix left, column vector right

Determinants: Signed Area

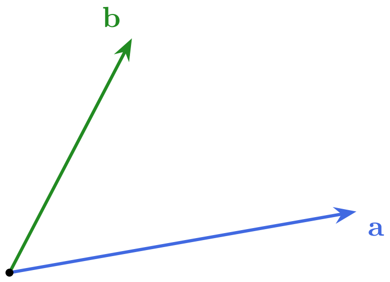
Multiply two vectors, get an area

Determinant: area between two arrows



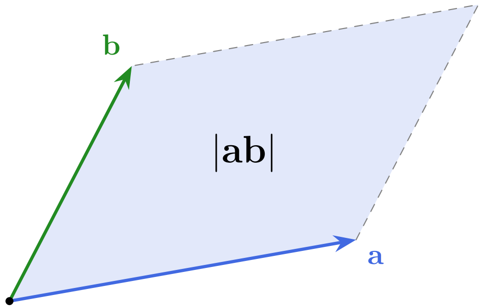
- Two arrows, one shared tail

Determinant: area between two arrows



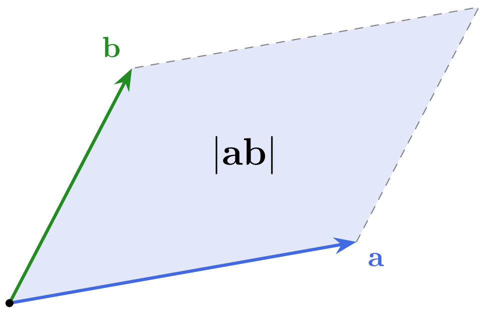
- Two arrows, one shared tail

Determinant: area between two arrows



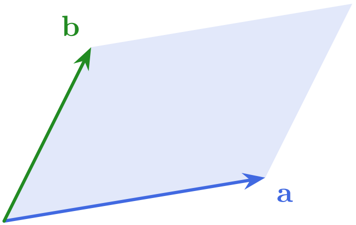
- Two arrows, one shared tail
- Area of the parallelogram

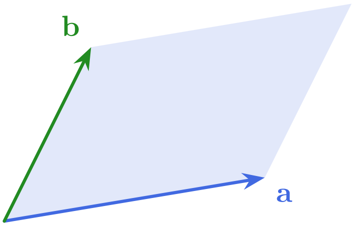
Determinant: area between two arrows



- Two arrows, one shared tail
- Area of the parallelogram
- Another way to multiply vectors

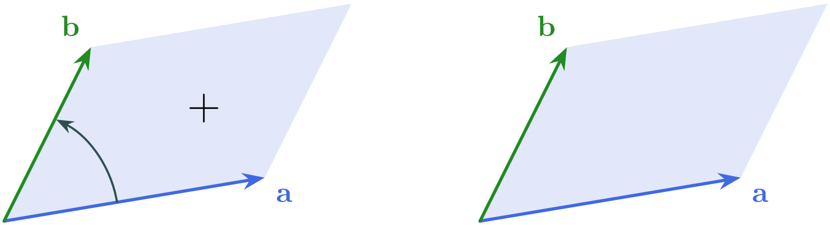
The sign says which way you turn





$|\mathbf{ab}|$ $|\mathbf{ba}|$

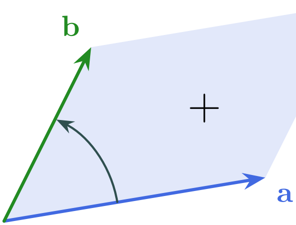
The sign says which way you turn



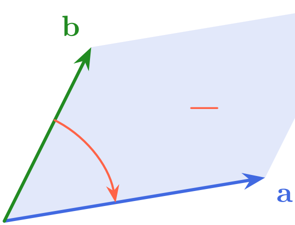
$|\mathbf{ab}|$ $|\mathbf{ba}|$

First to second counterclockwise: plus

The sign says which way you turn



$|\mathbf{ab}|$

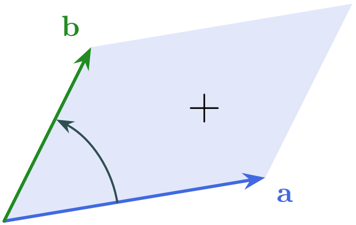


$|\mathbf{ba}|$

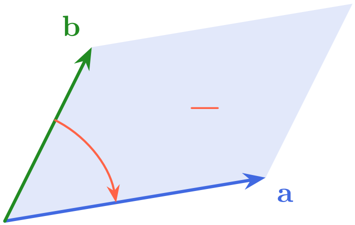
First to second counterclockwise: plus

Swap the order: minus

The sign says which way you turn



$|\mathbf{ab}|$



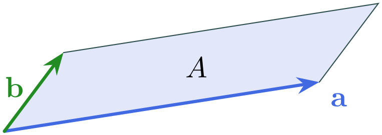
$|\mathbf{ba}|$

$$|\mathbf{ab}| = -|\mathbf{ba}|$$

First to second counterclockwise: plus

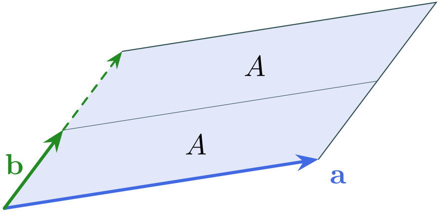
Swap the order: minus

Stretch one side, stretch the area



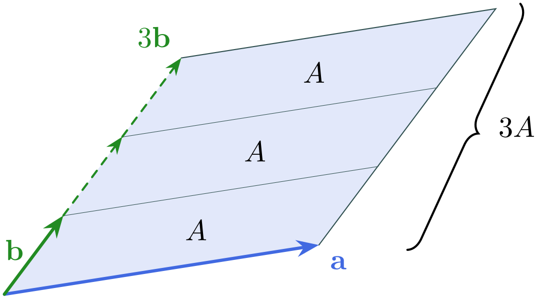
- Start with area A

Stretch one side, stretch the area



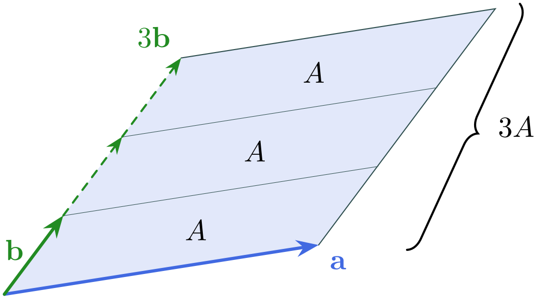
- Start with area A

Stretch one side, stretch the area



- Start with area A
- Triple one side, triple the area

Stretch one side, stretch the area



$$|(k\mathbf{a})\mathbf{b}| = |\mathbf{a}(k\mathbf{b})| = k|\mathbf{ab}|$$

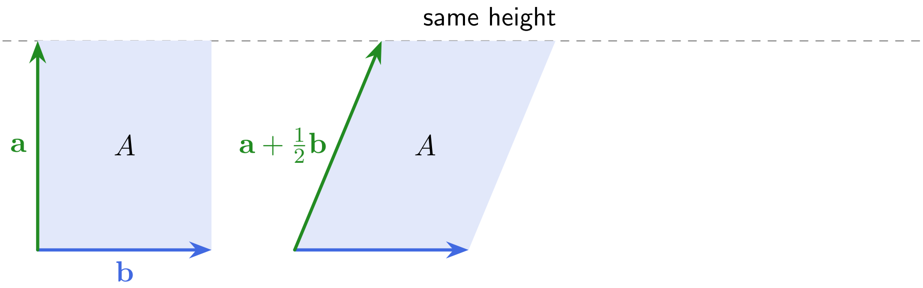
- Start with area A
- Triple one side, triple the area
- Works for any factor k

Sliding the top edge keeps the area

same height

Same base

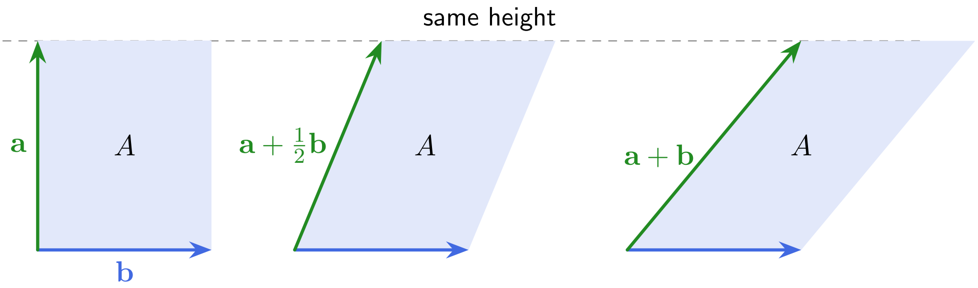
Sliding the top edge keeps the area



Same base

Slide the top edge

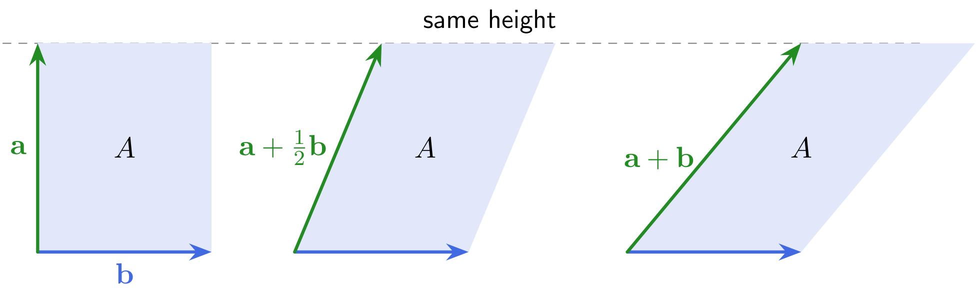
Sliding the top edge keeps the area



Same base

Slide the top edge

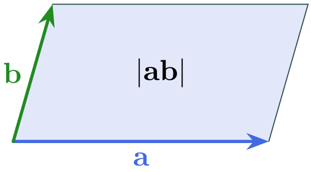
Sliding the top edge keeps the area



$$|(\mathbf{a} + k\mathbf{b})\mathbf{b}| = |\mathbf{a}(\mathbf{b} + k\mathbf{a})| = |\mathbf{ab}|$$

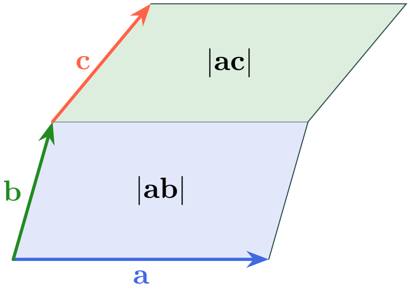
Same base | Slide the top edge | Area does not change

Two areas slide into one



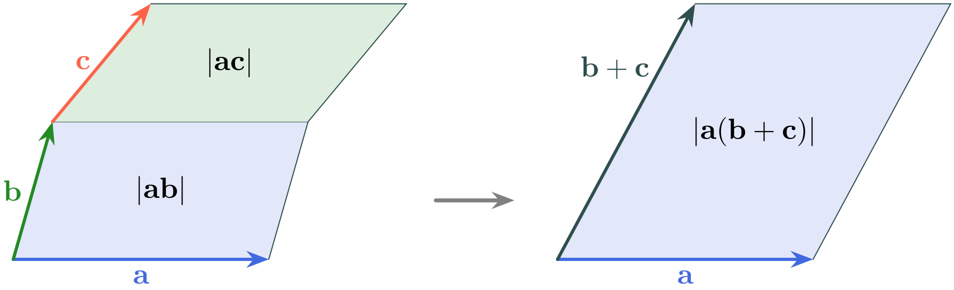
Two boxes on one base

Two areas slide into one



Two boxes on one base

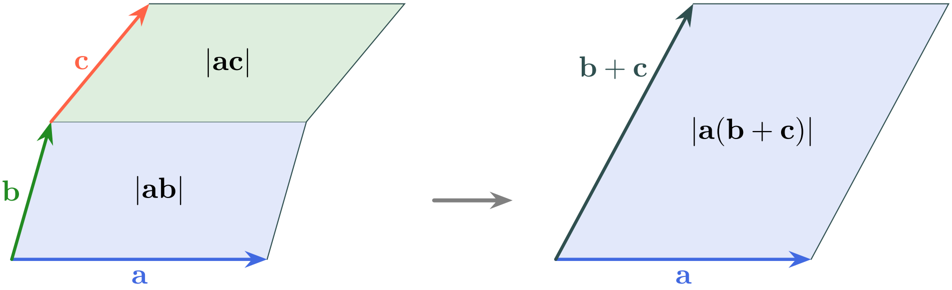
Two areas slide into one



Two boxes on one base

Slide into one box

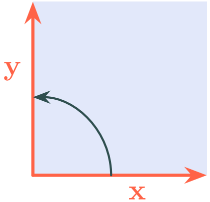
Two areas slide into one



$$|\mathbf{a}(\mathbf{b}+\mathbf{c})| = |\mathbf{ab}| + |\mathbf{ac}|$$

Two boxes on one base | Slide into one box | Areas add

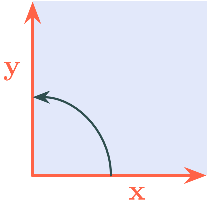
Unit arrows give one, minus one, zero



$$|\mathbf{xy}| = +1$$

Counterclockwise turn

Unit arrows give one, minus one, zero



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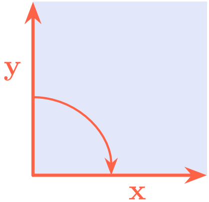
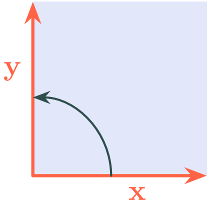
Counterclockwise turn



$$|\mathbf{yx}| = -1$$

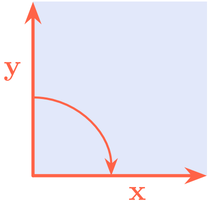
Reverse the turn

Unit arrows give one, minus one, zero



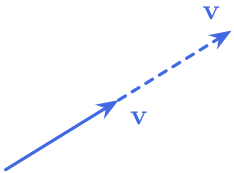
$$|\mathbf{xy}| = +1$$

Counterclockwise turn



$$|\mathbf{yx}| = -1$$

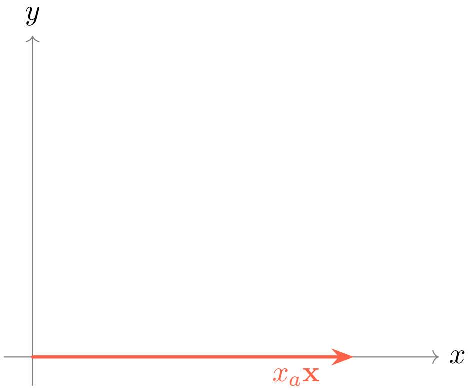
Reverse the turn



$$|\mathbf{vv}| = 0$$

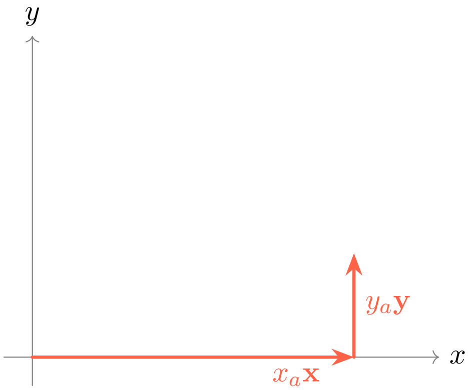
Same direction: flat, no area

Write each arrow using $\mathbf{x}$ and $\mathbf{y}$



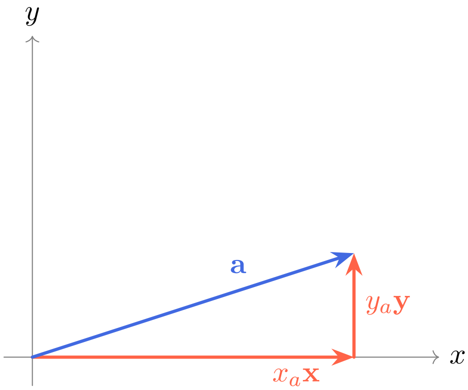
- Walk along x

Write each arrow using $\mathbf{x}$ and $\mathbf{y}$



- Walk along x
- Then walk along y

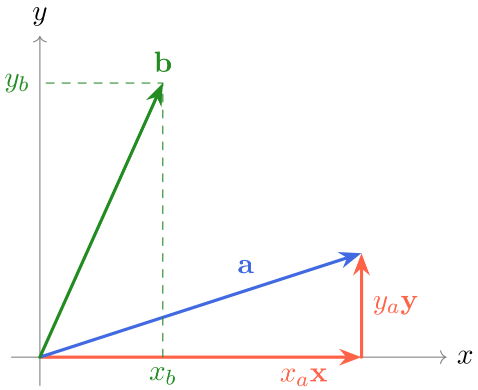
Write each arrow using $\mathbf{x}$ and $\mathbf{y}$



$$\mathbf{a} = x_a\mathbf{x} + y_a\mathbf{y}$$

- Walk along x
- Then walk along y

Write each arrow using $\mathbf{x}$ and $\mathbf{y}$



$$\mathbf{a} = x_a\mathbf{x} + y_a\mathbf{y}$$

$$\mathbf{b} = x_b\mathbf{x} + y_b\mathbf{y}$$

- Walk along x
- Then walk along y
- Every arrow splits this way

Expand, and most terms vanish

$$|\mathbf{xx}| = 0 \qquad |\mathbf{xy}| = +1 \qquad |\mathbf{yx}| = -1 \qquad |\mathbf{yy}| = 0$$

$$|\mathbf{ab}| = |(x_a\mathbf{x} + y_a\mathbf{y})(x_b\mathbf{x} + y_b\mathbf{y})|$$

components

Expand, and most terms vanish

$$|\mathbf{xx}| = 0 \qquad |\mathbf{xy}| = +1 \qquad |\mathbf{yx}| = -1 \qquad |\mathbf{yy}| = 0$$

$$\begin{aligned} |\mathbf{ab}| &= |(x_a\mathbf{x} + y_a\mathbf{y})(x_b\mathbf{x} + y_b\mathbf{y})| && \text{components} \\ &= x_a x_b|\mathbf{xx}| + x_a y_b|\mathbf{xy}| + y_a x_b|\mathbf{yx}| + y_a y_b|\mathbf{yy}| && \text{add, stretch rules} \end{aligned}$$

Multiply out every pair

Expand, and most terms vanish

$$|\mathbf{xx}| = 0 \qquad |\mathbf{xy}| = +1 \qquad |\mathbf{yx}| = -1 \qquad |\mathbf{yy}| = 0$$

$$\begin{aligned}
|\mathbf{ab}| &= |(x_a\mathbf{x} + y_a\mathbf{y})(x_b\mathbf{x} + y_b\mathbf{y})| && \text{components} \\
&= x_a x_b|\mathbf{xx}| + x_a y_b|\mathbf{xy}| + y_a x_b|\mathbf{yx}| + y_a y_b|\mathbf{yy}| && \text{add, stretch rules} \\
&= x_a x_b(0) + x_a y_b(+1) + y_a x_b(-1) + y_a y_b(0) && \text{unit arrow areas}
\end{aligned}$$

Multiply out every pair

Expand, and most terms vanish

$$|\mathbf{xx}| = 0 \qquad |\mathbf{xy}| = +1 \qquad |\mathbf{yx}| = -1 \qquad |\mathbf{yy}| = 0$$

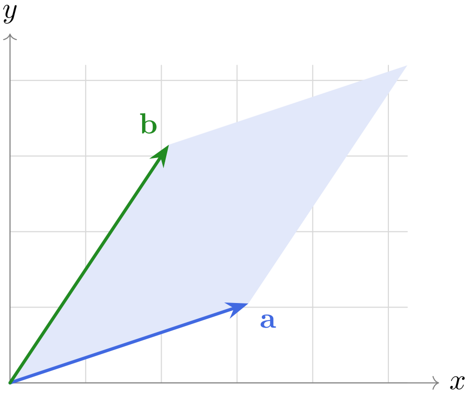
$$\begin{aligned}
|\mathbf{ab}| &= |(x_a\mathbf{x} + y_a\mathbf{y})(x_b\mathbf{x} + y_b\mathbf{y})| && \text{components} \\
&= x_a x_b|\mathbf{xx}| + x_a y_b|\mathbf{xy}| + y_a x_b|\mathbf{yx}| + y_a y_b|\mathbf{yy}| && \text{add, stretch rules} \\
&= x_a x_b(0) + x_a y_b(+1) + y_a x_b(-1) + y_a y_b(0) && \text{unit arrow areas} \\
&= x_a y_b - y_a x_b && \text{zeros drop out}
\end{aligned}$$

$$\boxed{|\mathbf{ab}| = x_a y_b - y_a x_b}$$

Multiply out every pair

Only two terms survive

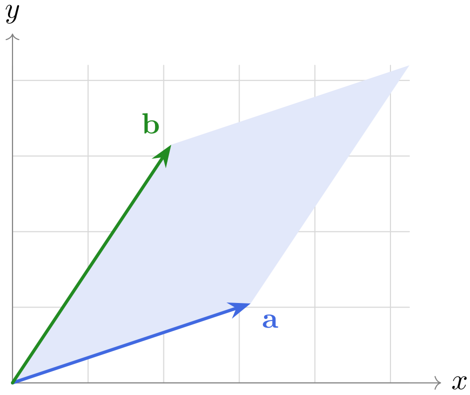
Worked numbers: the area is seven



$$\mathbf{a} = (3, 1), \quad \mathbf{b} = (2, 3)$$

$$|\mathbf{ab}| = x_a y_b - y_a x_b$$

Worked numbers: the area is seven

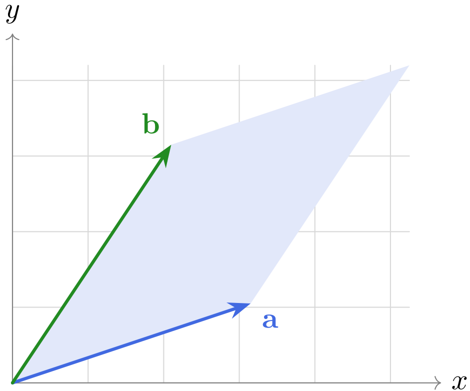


$$\mathbf{a} = (3, 1), \quad \mathbf{b} = (2, 3)$$

$$|\mathbf{ab}| = x_a y_b - y_a x_b$$
$$= (3)(3) - (1)(2)$$

- Plug in the numbers

Worked numbers: the area is seven

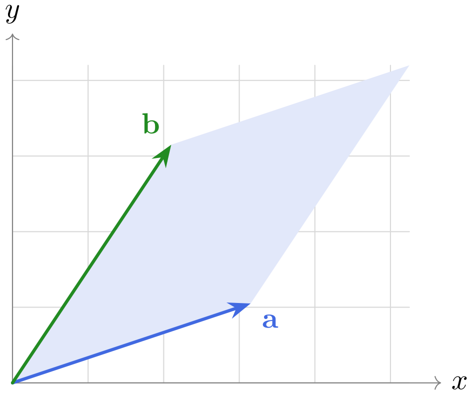


$$\mathbf{a} = (3,1), \quad \mathbf{b} = (2,3)$$

$$\begin{aligned} |\mathbf{ab}| &= x_a y_b - y_a x_b \\ &= (3)(3) - (1)(2) \\ &= 9 - 2 \end{aligned}$$

- Plug in the numbers

Worked numbers: the area is seven

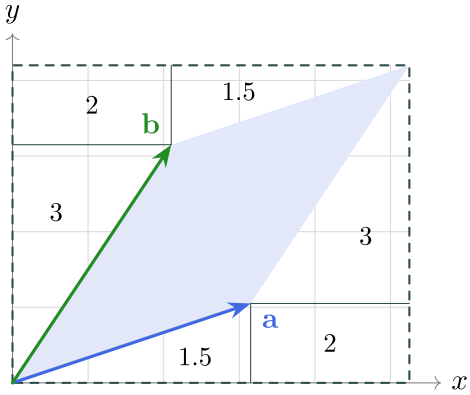


$$\mathbf{a} = (3,1), \quad \mathbf{b} = (2,3)$$

$$\begin{aligned} |\mathbf{ab}| &= x_a y_b - y_a x_b \\ &= (3)(3) - (1)(2) \\ &= 9 - 2 \\ &= 7 \end{aligned}$$

- Plug in the numbers

Worked numbers: the area is seven



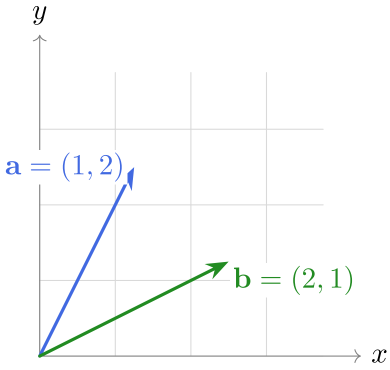
$$\mathbf{a} = (3, 1), \quad \mathbf{b} = (2, 3)$$

$$\begin{aligned} |\mathbf{ab}| &= x_a y_b - y_a x_b \\ &= (3)(3) - (1)(2) \\ &= 9 - 2 \\ &= 7 \end{aligned}$$

$$20 - (1.5 + 2 + 3 + 1.5 + 2 + 3) = 20 - 13 = 7$$

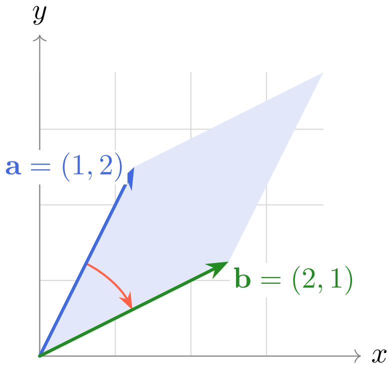
- Plug in the numbers
- Check: box minus corners

Quick check: find the area and its sign



- Compute $|\mathbf{ab}|$
- Plus or minus? Why?

Quick check: find the area and its sign



- Compute $|\mathbf{ab}|$
- Plus or minus? Why?

$$
\begin{aligned}
|\mathbf{ab}| &= (1)(1) - (2)(2) \\
&= 1 - 4 \\
&= -3
\end{aligned}
$$

- Minus: the turn is clockwise

Recap: the 2D determinant

$$|\mathbf{ab}| = x_a y_b - y_a x_b$$

Matrix Arithmetic

Rules for grids of numbers

A matrix is a grid of numbers

$$\begin{array}{cc} & \begin{array}{ccc} \text{col 1} & \text{col 2} & \end{array} \\ \begin{array}{c} \text{row 1} \\ \end{array} & \begin{bmatrix} 1.7 & -1.2 & 4.2 \\ 3.0 & 4.5 & -7.2 \end{bmatrix} \end{array}$$

2×3 matrix

Rows go across

A matrix is a grid of numbers

$$\begin{array}{c c} & \begin{array}{ccc} \text{col 1} & \text{col 2} & \end{array} \\ \begin{array}{c} \text{row 1} \\ \text{row 2} \end{array} & \begin{bmatrix} 1.7 & -1.2 & 4.2 \\ 3.0 & 4.5 & -7.2 \end{bmatrix} \end{array}$$

2×3 matrix

Rows go across

A matrix is a grid of numbers

$$
\begin{array}{cc}
 & \begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 3} \end{array} \\
\begin{array}{c} \text{row 1} \\ \text{row 2} \end{array} & \begin{bmatrix} 1.7 & -1.2 & 4.2 \\ 3.0 & 4.5 & -7.2 \end{bmatrix}
\end{array}
$$

2×3 matrix

Rows go across

Columns go down

A matrix is a grid of numbers

$$
\begin{array}{c c}
 & \begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 3} \end{array} \\
\begin{array}{c} \text{row 1} \\ \text{row 2} \end{array} & \begin{bmatrix} 1.7 & -1.2 & 4.2 \\ 3.0 & 4.5 & -7.2 \end{bmatrix}
\end{array}
$$

a_{23}: row 2, column 3

2×3 matrix

Rows go across

Columns go down

Row first, then column

Scale and add, one entry at a time

$$2\begin{bmatrix}1 & -4\\ 3 & 2\end{bmatrix} = \begin{bmatrix}2 & -8\\ 6 & 4\end{bmatrix}$$

Number times matrix: every entry

Scale and add, one entry at a time

$$2\begin{bmatrix}1 & -4\\ 3 & 2\end{bmatrix} = \begin{bmatrix}2 & -8\\ 6 & 4\end{bmatrix}$$

$$2\cdot(-4) = -8$$

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$$2\begin{bmatrix}1 & -4\\3 & 2\end{bmatrix} = \begin{bmatrix}2 & -8\\6 & 4\end{bmatrix}$$

$$2\cdot(-4) = -8$$

$$\begin{bmatrix}1 & -4\\3 & 2\end{bmatrix} + \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix} = \begin{bmatrix}3 & -2\\5 & 4\end{bmatrix}$$

Number times matrix: every entry

Plus: entries in the same spot

Scale and add, one entry at a time

$$2\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ 6 & 4 \end{bmatrix}$$

$$2 \cdot (-4) = -8$$

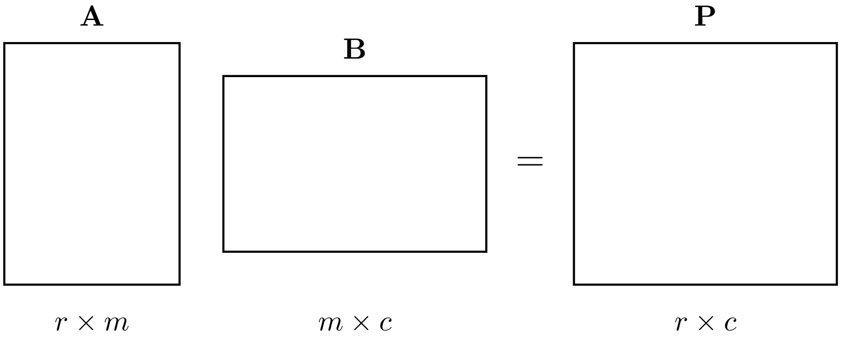
$$\begin{bmatrix} 1 & -4 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$$

$$(-4) + 2 = -2$$

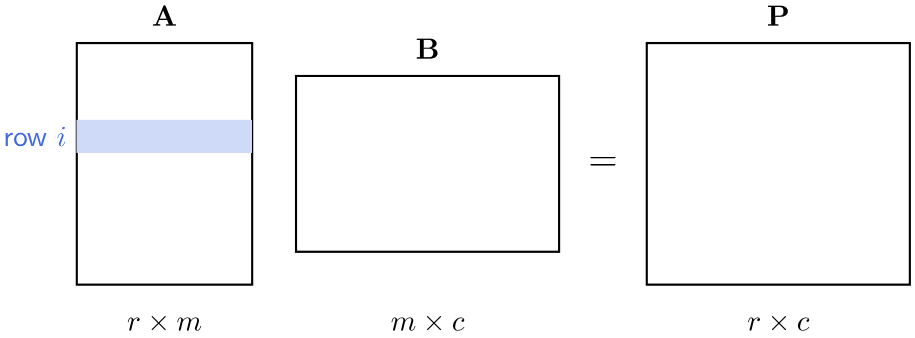
Number times matrix: every entry

Plus: entries in the same spot

Left row meets right column

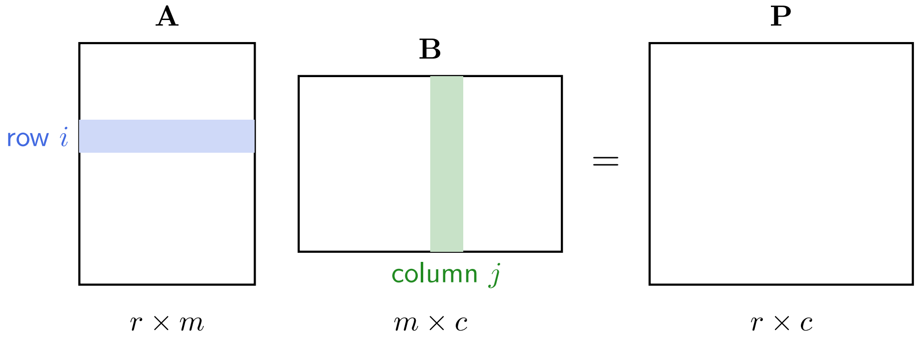


Left row meets right column



Pick row i on the left

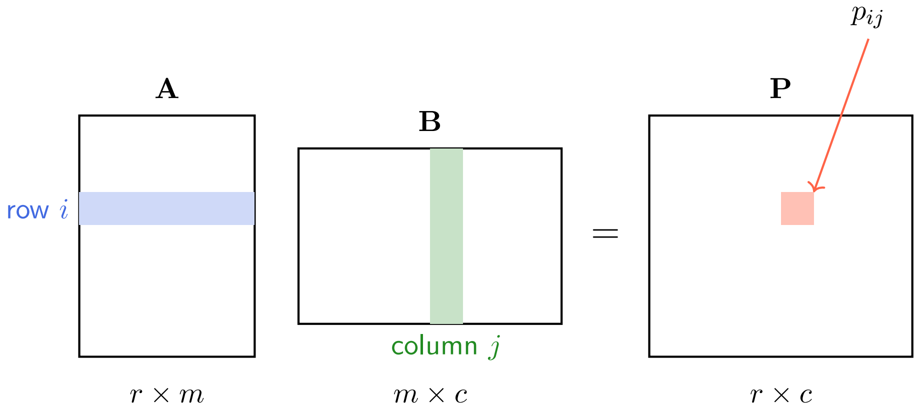
Left row meets right column



Pick row i on the left

Pick column j on the right

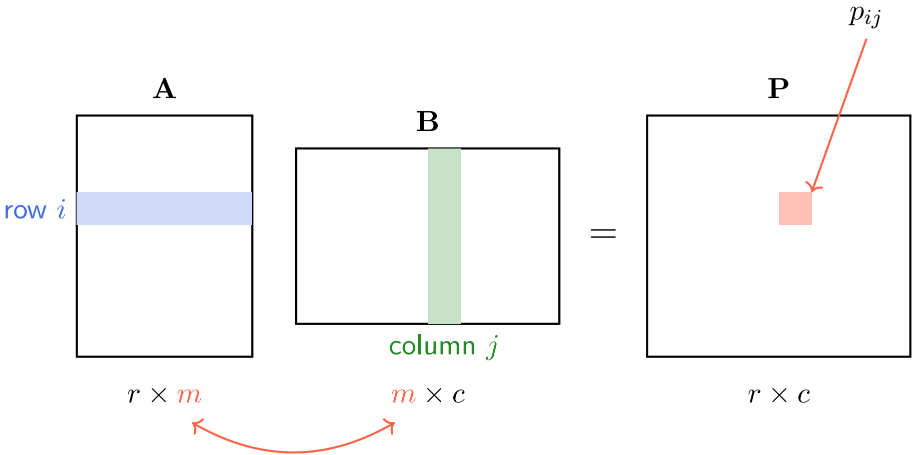
Left row meets right column



$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

Pick row i on the left   Pick column j on the right

Left row meets right column



$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

Pick row i on the left | Pick column j on the right | Inner sizes must match

Worked product: one row at a time

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} & & & \\ & & & \\ & & & \end{bmatrix}$$

3×2 $\quad$ 2×4 $\quad$ 3×4

Inner sizes match: 2 and 2

Answer: 3 rows, 4 columns

Worked product: one row at a time

$$\underset{3 \times 2}{\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}} \underset{2 \times 4}{\begin{bmatrix} 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 \end{bmatrix}} = \underset{3 \times 4}{\begin{bmatrix} 0 & 1 & 2 & 3 \\ & & & \\ & & & \end{bmatrix}}$$

row 1: $0 \cdot 6 + 1 \cdot 0 = 0 \quad 0 \cdot 7 + 1 \cdot 1 = 1 \quad 0 \cdot 8 + 1 \cdot 2 = 2 \quad 0 \cdot 9 + 1 \cdot 3 = 3$

Inner sizes match: 2 and 2

Answer: 3 rows, 4 columns

Worked product: one row at a time

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 12 & 17 & 22 & 27 \\ & & & \end{bmatrix}$$

$$3 \times 2 \qquad 2 \times 4 \qquad 3 \times 4$$

row 1: $0 \cdot 6 + 1 \cdot 0 = 0$ $\quad 0 \cdot 7 + 1 \cdot 1 = 1$ $\quad 0 \cdot 8 + 1 \cdot 2 = 2$ $\quad 0 \cdot 9 + 1 \cdot 3 = 3$

row 2: $2 \cdot 6 + 3 \cdot 0 = 12$ $\quad 2 \cdot 7 + 3 \cdot 1 = 17$ $\quad 2 \cdot 8 + 3 \cdot 2 = 22$ $\quad 2 \cdot 9 + 3 \cdot 3 = 27$

Inner sizes match: 2 and 2

Answer: 3 rows, 4 columns

Worked product: one row at a time

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ {\color{blue}4} & {\color{blue}5} \end{bmatrix} \begin{bmatrix} 6 & 7 & 8 & 9 \\ 0 & 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 12 & 17 & 22 & 27 \\ 24 & 33 & 42 & 51 \end{bmatrix}$$

3×2 $\quad$ 2×4 $\quad$ 3×4

row 1: $0 \cdot 6 + 1 \cdot 0 = 0$ $\quad 0 \cdot 7 + 1 \cdot 1 = 1$ $\quad 0 \cdot 8 + 1 \cdot 2 = 2$ $\quad 0 \cdot 9 + 1 \cdot 3 = 3$

row 2: $2 \cdot 6 + 3 \cdot 0 = 12$ $\quad 2 \cdot 7 + 3 \cdot 1 = 17$ $\quad 2 \cdot 8 + 3 \cdot 2 = 22$ $\quad 2 \cdot 9 + 3 \cdot 3 = 27$

row 3: $4 \cdot 6 + 5 \cdot 0 = 24$ $\quad 4 \cdot 7 + 5 \cdot 1 = 33$ $\quad 4 \cdot 8 + 5 \cdot 2 = 42$ $\quad 4 \cdot 9 + 5 \cdot 3 = 51$

Inner sizes match: 2 and 2

Answer: 3 rows, 4 columns

Swap the order, get a different answer

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Swap the order, get a different answer

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Swap the order, get a different answer

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 1 \\ 1 \cdot 1 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Swap the order, get a different answer

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 1 \cdot 1 + 1 \cdot 1 & 1 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 1 + 1 \cdot 1 & 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\mathbf{BA} = \begin{bmatrix} 1 \cdot 1 + 0 \cdot 0 & 1 \cdot 1 + 0 \cdot 1 \\ 1 \cdot 1 + 1 \cdot 0 & 1 \cdot 1 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\mathbf{AB} \neq \mathbf{BA} \qquad (\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

Order changes the answer

Grouping does not

Identity: the matrix that acts like one

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Square matrices only

Identity: the matrix that acts like one

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonal

- Ones down the diagonal
- Zeros everywhere else
- Square matrices only

Identity: the matrix that acts like one

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diagonal

$$x \cdot \frac{1}{x} = 1 \quad \longleftrightarrow \quad \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

- Ones down the diagonal
- Zeros everywhere else
- The "matrix one"
- Square matrices only

The inverse undoes: the product is $\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$$

Times its inverse gives $\mathbf{I}$

The inverse undoes: the product is $\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & \\ & \end{bmatrix}$$

row 1, col 1: $\quad 1(-2.0) + 2(1.5) = -2 + 3 = 1$

Times its inverse gives $\mathbf{I}$

The inverse undoes: the product is $\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ & \end{bmatrix}$$

row 1, col 1: $1(-2.0) + 2(1.5) = -2 + 3 = 1$

row 1, col 2: $1(1.0) + 2(-0.5) = 1 - 1 = 0$

Times its inverse gives $\mathbf{I}$

The inverse undoes: the product is $\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \end{bmatrix}$$

row 1, col 1: $1(-2.0) + 2(1.5) = -2 + 3 = 1$

row 1, col 2: $1(1.0) + 2(-0.5) = 1 - 1 = 0$

row 2, col 1: $3(-2.0) + 4(1.5) = -6 + 6 = 0$

Times its inverse gives $\mathbf{I}$

The inverse undoes: the product is $\mathbf{I}$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2.0 & 1.0 \\ 1.5 & -0.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

row 1, col 1: $1(-2.0) + 2(1.5) = -2 + 3 = 1$

row 1, col 2: $1(1.0) + 2(-0.5) = 1 - 1 = 0$

row 2, col 1: $3(-2.0) + 4(1.5) = -6 + 6 = 0$

row 2, col 2: $3(1.0) + 4(-0.5) = 3 - 2 = 1$

Times its inverse gives $\mathbf{I}$

Both orders: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

Transpose turns rows into columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Transpose turns rows into columns

$$\begin{bmatrix} \boxed{1 \quad 2} \\ 3 \quad 4 \\ 5 \quad 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Row 1 becomes column 1

Transpose turns rows into columns

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Row 1 becomes column 1 Row 2 becomes column 2

Transpose turns rows into columns

$$\begin{bmatrix}1 & 2\\ 3 & 4\\ 5 & 6\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}1 & 3 & 5\\ 2 & 4 & 6\end{bmatrix} \qquad a_{ij} = a'_{ji}$$

Row 1 becomes column 1 Row 2 becomes column 2 Indices swap places

Undo a product in reverse order

$$\underbrace{\boxed{\mathbf{A}}\;\boxed{\mathbf{B}}}_{\mathbf{AB}}\quad\underbrace{\boxed{\mathbf{B}^{-1}}\;\boxed{\mathbf{A}^{-1}}}_{\mathbf{B}^{-1}\mathbf{A}^{-1}}$$

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1})$$

Undo a product in reverse order

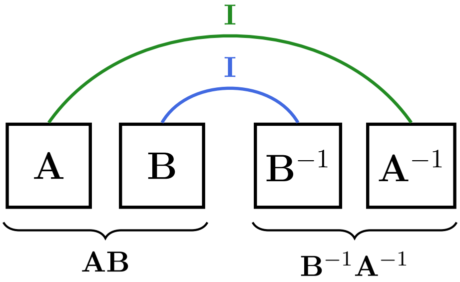
$$\mathbf{I}$$

$$\underbrace{\boxed{\mathbf{A}}\ \boxed{\mathbf{B}}}_{\mathbf{AB}}\quad \underbrace{\boxed{\mathbf{B}^{-1}}\ \boxed{\mathbf{A}^{-1}}}_{\mathbf{B}^{-1}\mathbf{A}^{-1}}$$

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} \qquad \text{regroup}$$

Inner pair cancels first

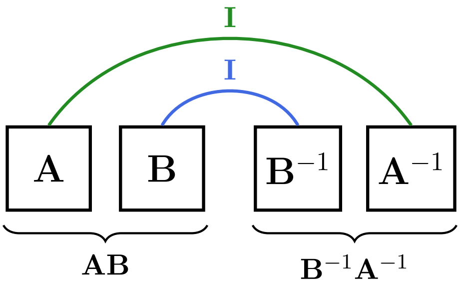
Undo a product in reverse order



$$
\begin{aligned}
(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) &= \mathbf{A}(\mathbf{B}\mathbf{B}^{-1})\mathbf{A}^{-1} && \text{regroup} \\
&= \mathbf{A}\,\mathbf{I}\,\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} && \text{inner pair first}
\end{aligned}
$$

Inner pair cancels first

Undo a product in reverse order

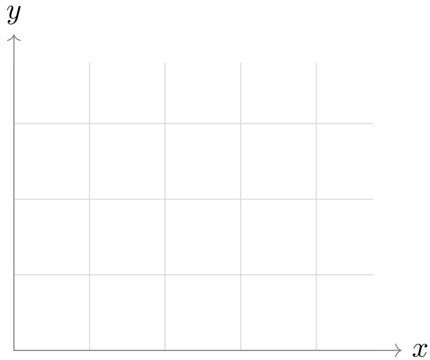


$$
\begin{aligned}
(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) &= \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} && \text{regroup} \\
&= \mathbf{A}\,\mathbf{I}\,\mathbf{A}^{-1} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I} && \text{inner pair first}
\end{aligned}
$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \qquad (\mathbf{AB})^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$$

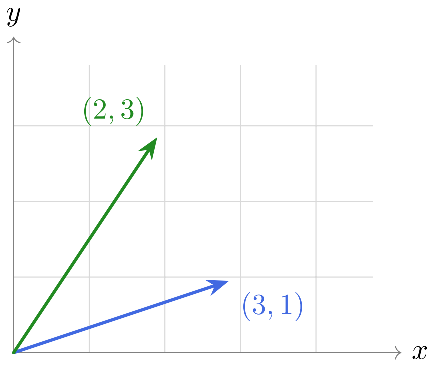
Inner pair cancels first | Inverse: reverse the order | Transpose: reverse too

A matrix's determinant uses its columns



$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$$

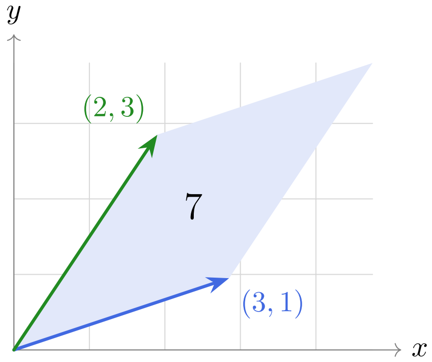
A matrix's determinant uses its columns



$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$$

- Columns are two arrows

A matrix's determinant uses its columns

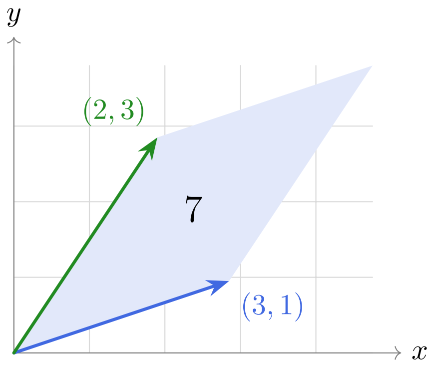


$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$$

$$|\mathbf{A}| = (3)(3) - (1)(2) = 9 - 2 = 7$$

- Columns are two arrows
- Same area as before

A matrix's determinant uses its columns



$$\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}$$

$$|\mathbf{A}| = (3)(3) - (1)(2) = 9 - 2 = 7$$

$$|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$$

$$|\mathbf{A}^{-1}| = \frac{1}{|\mathbf{A}|} \qquad |\mathbf{A}^{\mathrm{T}}| = |\mathbf{A}|$$

- Columns are two arrows
- Same area as before

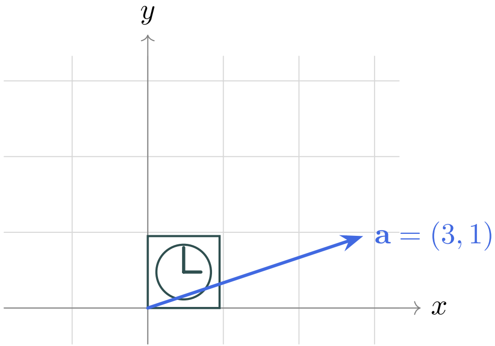
Recap: multiplying matrices

$$p_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{im}b_{mj}$$

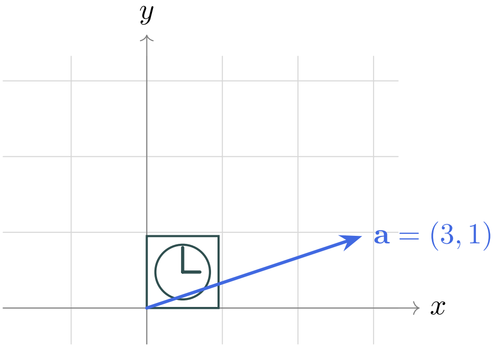
Vectors in Matrix Form

A vector is a one-column matrix

Column vector, matrix on its left



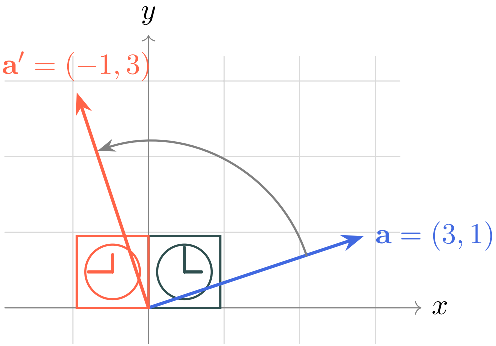
Column vector, matrix on its left



$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} -y_a \\ x_a \end{bmatrix}$$

- Vector stands up: 2×1

Column vector, matrix on its left

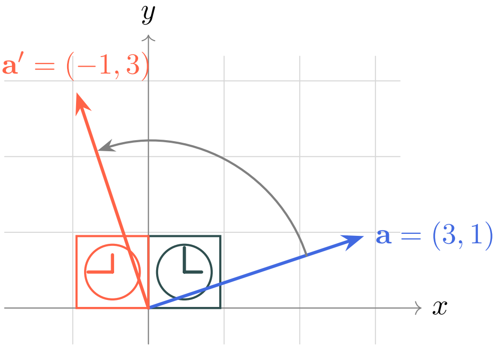


$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} -y_a \\ x_a \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 3 + (-1) \cdot 1 \\ 1 \cdot 3 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

- Vector stands up: 2×1
- Turns 90° counterclockwise

Column vector, matrix on its left



$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} -y_a \\ x_a \end{bmatrix}$$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \cdot 3 + (-1) \cdot 1 \\ 1 \cdot 3 + 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 3 \end{bmatrix}$$

- Vector stands up: 2×1
- Turns 90° counterclockwise
- Row style needs the transpose

Row view: dot each row with $\mathbf{x}$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{x}}$$

Row view: dot each row with $\mathbf{x}$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{array}{c} \mathbf{r}_1 \\ \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \end{array}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{x}}$$

Take one row

Row view: dot each row with $\mathbf{x}$

$$\underbrace{\begin{bmatrix} 8 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{array}{c} \mathbf{r}_1 \\ \begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix} \end{array}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{x}}$$

$$y_1 = \mathbf{r}_1 \cdot \mathbf{x} = (3)(2) + (2)(1) = 6 + 2 = 8$$

Take one row

Dot it with $\mathbf{x}$

Row view: dot each row with $\mathbf{x}$

$$\underbrace{\begin{bmatrix} 8 \\ 5 \end{bmatrix}}_{\mathbf{y}} = \begin{array}{c} {\color{blue}\mathbf{r}_1} \\ \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}}_{\mathbf{A}} \\ {\color{green}\mathbf{r}_2} \end{array} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{x}}$$

$$y_1 = \mathbf{r}_1 \cdot \mathbf{x} = (3)(2) + (2)(1) = 6 + 2 = 8$$

$$y_2 = \mathbf{r}_2 \cdot \mathbf{x} = (1)(2) + (3)(1) = 2 + 3 = 5$$

Take one row | Dot it with $\mathbf{x}$ | One row, one output

Row view: dot each row with $\mathbf{x}$

$$\underbrace{\begin{bmatrix} 8 \\ 5 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 3 & 2 \\ 1 & 3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\mathbf{x}}$$

$\mathbf{r}_1$, $\mathbf{r}_2$

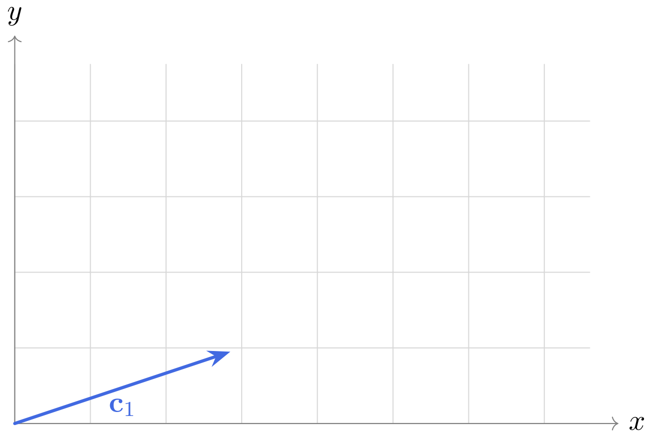
$$y_1 = \mathbf{r}_1 \cdot \mathbf{x} = (3)(2) + (2)(1) = 6 + 2 = 8$$

$$y_2 = \mathbf{r}_2 \cdot \mathbf{x} = (1)(2) + (3)(1) = 2 + 3 = 5$$

$$y_i = \mathbf{r}_i \cdot \mathbf{x} \qquad \mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b}$$

Take one row | Dot it with $\mathbf{x}$ | One row, one output

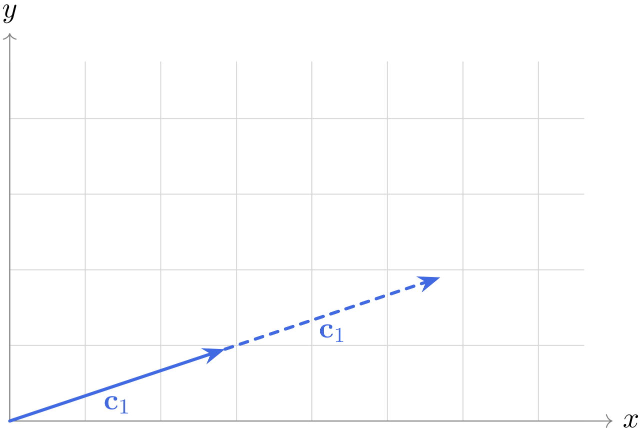
Column view: mix the columns



$$\mathbf{y} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2$$

- Columns: the ingredient arrows

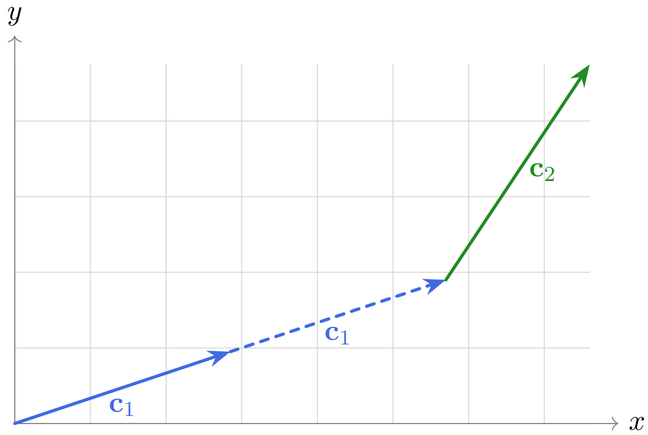
Column view: mix the columns



$$\mathbf{y} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2$$
$$= 2\begin{bmatrix}3\\1\end{bmatrix} + 1\begin{bmatrix}2\\3\end{bmatrix}$$

- Columns: the ingredient arrows
- $\mathbf{x}$ says how much of each

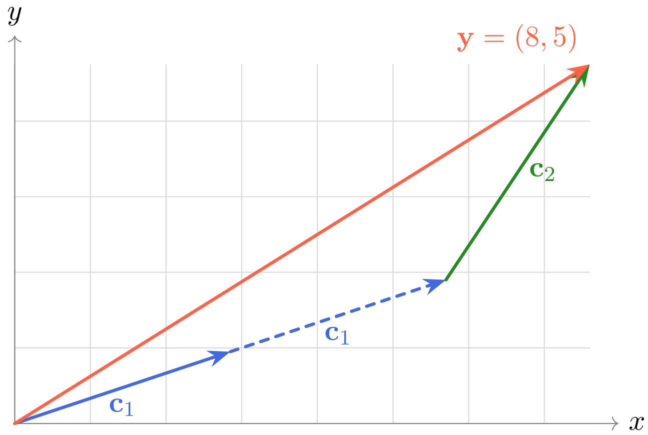
Column view: mix the columns



$$\begin{aligned}\mathbf{y} &= x_1\mathbf{c}_1 + x_2\mathbf{c}_2 \\ &= 2\begin{bmatrix}3\\1\end{bmatrix} + 1\begin{bmatrix}2\\3\end{bmatrix} \\ &= \begin{bmatrix}6\\2\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix}\end{aligned}$$

- Columns: the ingredient arrows
- $\mathbf{x}$ says how much of each

Column view: mix the columns



$$
\begin{aligned}
\mathbf{y} &= x_1\mathbf{c}_1 + x_2\mathbf{c}_2 \\
&= 2\begin{bmatrix}3\\1\end{bmatrix} + 1\begin{bmatrix}2\\3\end{bmatrix} \\
&= \begin{bmatrix}6\\2\end{bmatrix} + \begin{bmatrix}2\\3\end{bmatrix} \\
&= \begin{bmatrix}8\\5\end{bmatrix}
\end{aligned}
$$

- Columns: the ingredient arrows
- $\mathbf{x}$ says how much of each
- Same answer as the rows

Diagonal and symmetric: check the diagonal

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

diagonal

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 7 \\ 2 & 7 & 1 \end{bmatrix}$$

symmetric

Diagonal and symmetric: check the diagonal

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

diagonal

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 7 \\ 2 & 7 & 1 \end{bmatrix}$$

symmetric

Zeros off the diagonal

Diagonal and symmetric: check the diagonal

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 7 \\ 2 & 7 & 1 \end{bmatrix}$$

diagonal

symmetric

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}$$

Zeros off the diagonal

Same as its transpose

Diagonal and symmetric: check the diagonal

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

diagonal

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 9 & 7 \\ 2 & 7 & 1 \end{bmatrix}$$

symmetric

$$\mathbf{A}^{\mathrm{T}} = \mathbf{A}$$

Zeros off the diagonal

Same as its transpose

Mirror pairs match

Orthogonal: unit columns at right angles

$$\mathbf{R} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Orthogonal: unit columns at right angles

$$\begin{array}{cc} \mathbf{R} = & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ & \begin{array}{ccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{array} \end{array}$$

Columns have length one

Orthogonal: unit columns at right angles

$$\mathbf{R} = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{array}{ccc} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{array} \end{array} \qquad \mathrm{R}^{\mathrm{T}}\mathbf{R} = \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{c}_1 & \mathbf{c}_1 \cdot \mathbf{c}_2 & \mathbf{c}_1 \cdot \mathbf{c}_3 \\ \mathbf{c}_2 \cdot \mathbf{c}_1 & \mathbf{c}_2 \cdot \mathbf{c}_2 & \mathbf{c}_2 \cdot \mathbf{c}_3 \\ \mathbf{c}_3 \cdot \mathbf{c}_1 & \mathbf{c}_3 \cdot \mathbf{c}_2 & \mathbf{c}_3 \cdot \mathbf{c}_3 \end{bmatrix}$$

Columns have length one

Columns at right angles

Orthogonal: unit columns at right angles

$$\mathbf{R} = \begin{array}{c} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \\ \begin{matrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \end{matrix} \end{array}$$

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \begin{bmatrix} \mathbf{c}_1 \cdot \mathbf{c}_1 & \mathbf{c}_1 \cdot \mathbf{c}_2 & \mathbf{c}_1 \cdot \mathbf{c}_3 \\ \mathbf{c}_2 \cdot \mathbf{c}_1 & \mathbf{c}_2 \cdot \mathbf{c}_2 & \mathbf{c}_2 \cdot \mathbf{c}_3 \\ \mathbf{c}_3 \cdot \mathbf{c}_1 & \mathbf{c}_3 \cdot \mathbf{c}_2 & \mathbf{c}_3 \cdot \mathbf{c}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I} = \mathbf{R}\mathbf{R}^{\mathrm{T}} \qquad |\mathbf{R}| = +1 \text{ or } -1$$

Columns have length one Columns at right angles Transpose is the inverse

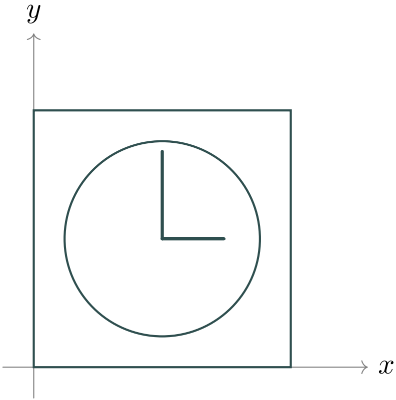
Recap: the column view

$$\mathbf{y} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2 + x_3\mathbf{c}_3$$

2D Linear Transformations

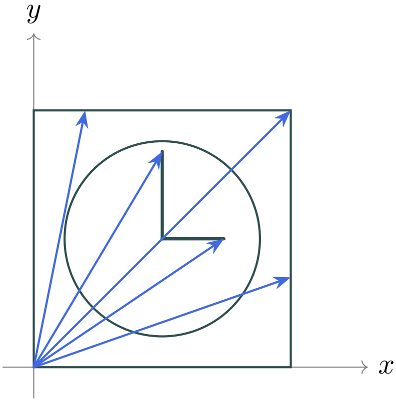
Four numbers move the clock

The clock is many arrows from the origin



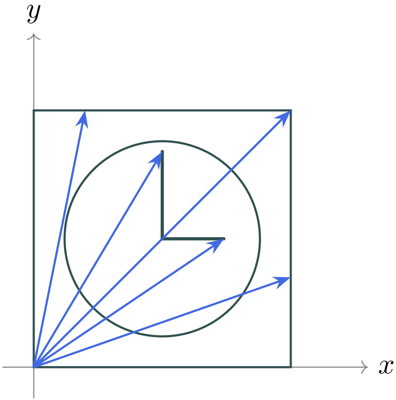
- Clock: a set of points

The clock is many arrows from the origin



- Clock: a set of points
- Each point: tip of an arrow

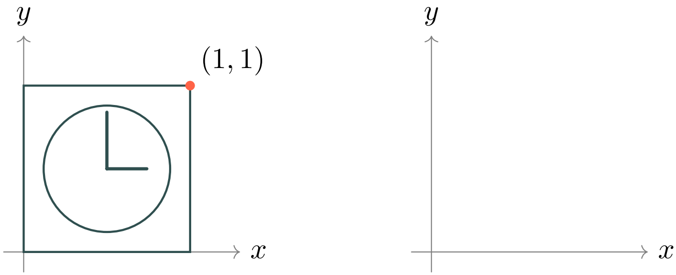
The clock is many arrows from the origin



$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{bmatrix}$$

- Clock: a set of points
- Each point: tip of an arrow
- Matrix moves every arrow

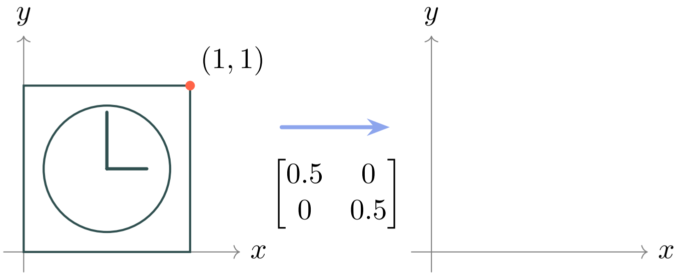
Scale: the factors sit on the diagonal



$$\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

- Factors on the diagonal

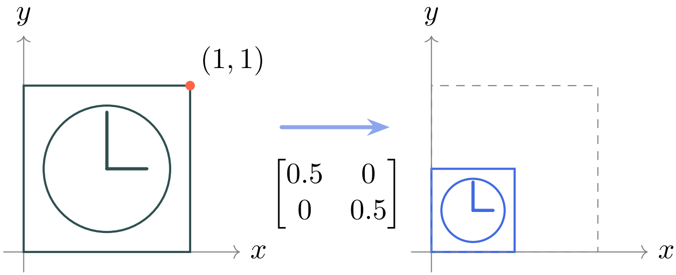
Scale: the factors sit on the diagonal



$$\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

- Factors on the diagonal
- Zeros everywhere else

Scale: the factors sit on the diagonal

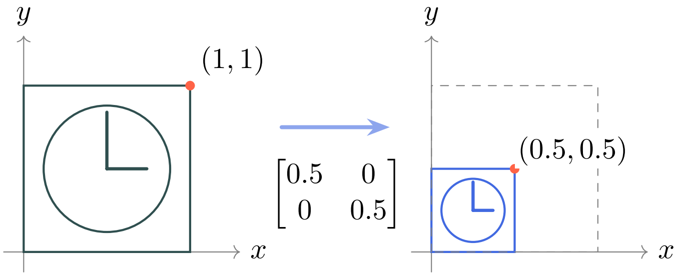


$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

- Factors on the diagonal
- Zeros everywhere else
- Half as wide, half as tall

Scale: the factors sit on the diagonal

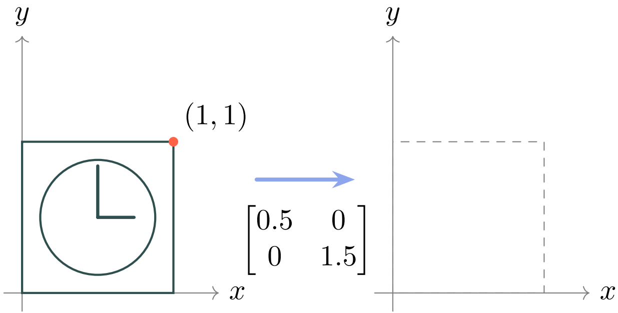


$$\text{scale}(s_x, s_y) = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$

$$\begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \cdot 1 + 0 \cdot 1 \\ 0 \cdot 1 + 0.5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

- Factors on the diagonal
- Zeros everywhere else
- Half as wide, half as tall

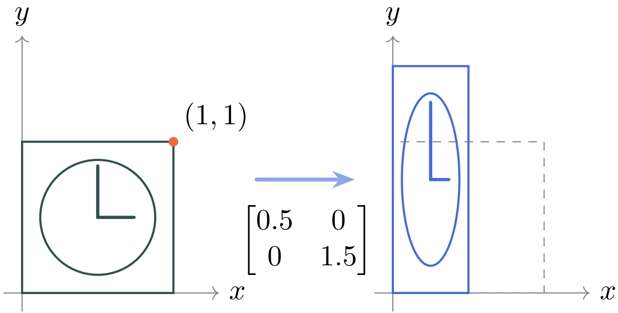
Unequal scales stretch circles into ellipses



$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

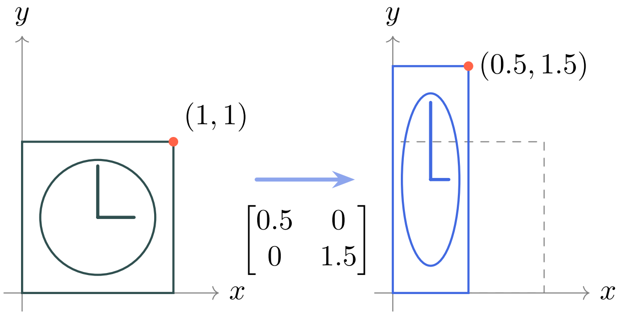
- Half as wide
- One and a half times as tall

Unequal scales stretch circles into ellipses



- Half as wide
- One and a half times as tall
- Square becomes a rectangle
- Circle becomes an ellipse

Unequal scales stretch circles into ellipses

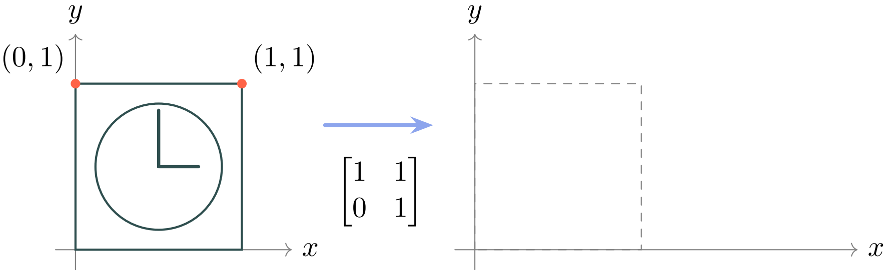


$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix}$$

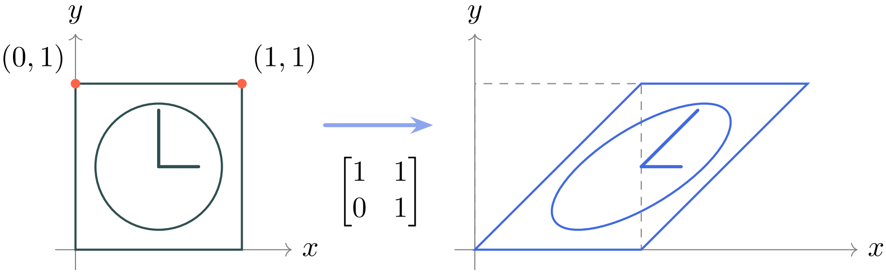
$$\begin{bmatrix} 0.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5 \cdot 1 + 0 \cdot 1 \\ 0 \cdot 1 + 1.5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \end{bmatrix}$$

- Half as wide
- One and a half times as tall
- Square becomes a rectangle
- Circle becomes an ellipse

Shear pushes sideways, more higher up

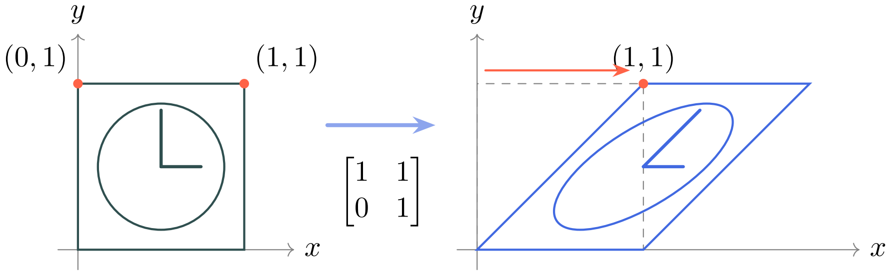


Shear pushes sideways, more higher up



Bottom edge stays put

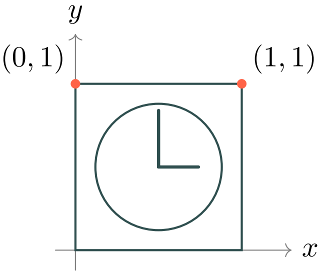
Shear pushes sideways, more higher up



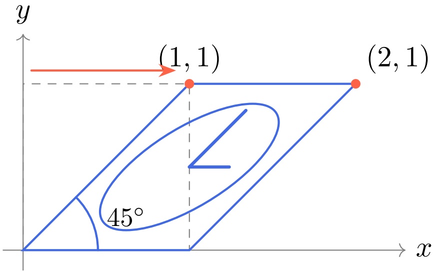
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \cdot 0 + 1 \cdot 1 \\ 0 \cdot 0 + 1 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Bottom edge stays put　　　　Top edge slides right

Shear pushes sideways, more higher up



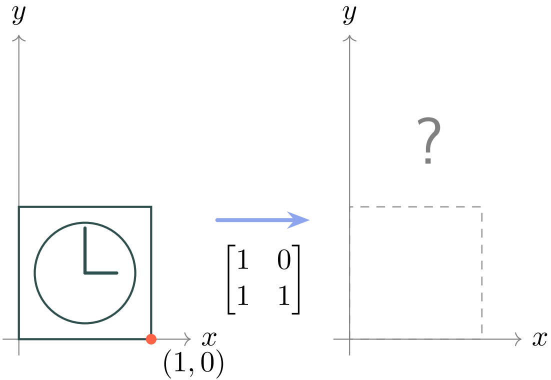
$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$$



$$\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\cdot 0+1\cdot 1\\0\cdot 0+1\cdot 1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \qquad \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\cdot 1+1\cdot 1\\0\cdot 1+1\cdot 1\end{bmatrix}=\begin{bmatrix}2\\1\end{bmatrix}$$

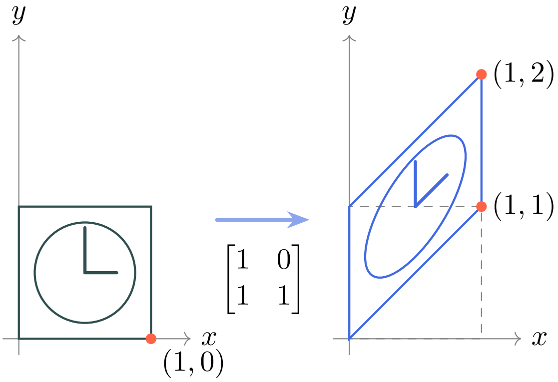
Bottom edge stays put Top edge slides right Upright lines lean 45°

Quick check: predict this shear



- Where does $(1, 0)$ go?
- Sketch the whole clock

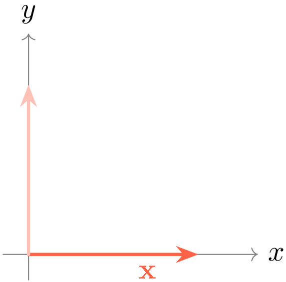
Quick check: predict this shear



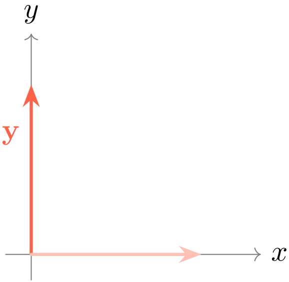
$$\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\cdot 1 + 0\cdot 0\\1\cdot 1 + 1\cdot 0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix}$$

- Where does $(1,0)$ go?
- Sketch the whole clock
- Right edge slides up

A shear tilts just one axis

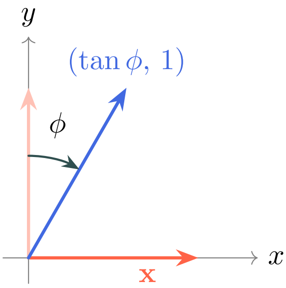


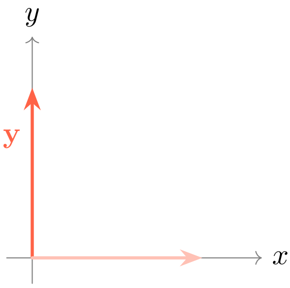
$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 1 & 0 \\ \tan\phi & 1 \end{bmatrix}$$

A shear tilts just one axis



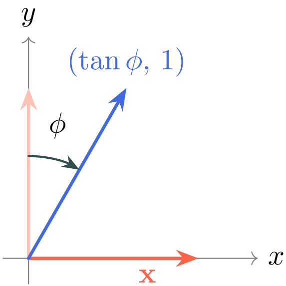


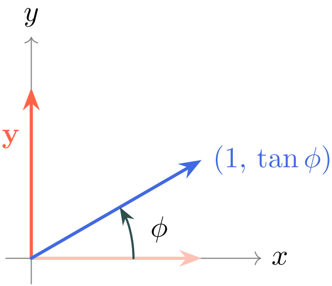
$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ \tan\phi & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + \tan\phi\cdot 1 \\ 0\cdot 0 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} \tan\phi \\ 1 \end{bmatrix}$$

Upright axis tilts clockwise

A shear tilts just one axis





$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix}$$

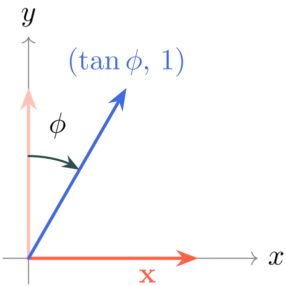
$$\begin{bmatrix} 1 & 0 \\ \tan\phi & 1 \end{bmatrix}$$

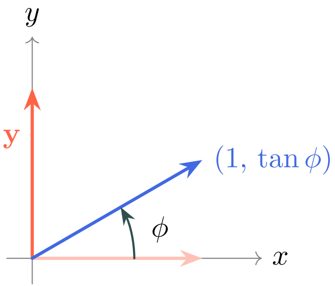
$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + \tan\phi \cdot 1 \\ 0\cdot 0 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} \tan\phi \\ 1 \end{bmatrix}$$

Upright axis tilts clockwise

Flat axis tilts up

A shear tilts just one axis





$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \tan\phi & 1 \end{bmatrix}$$

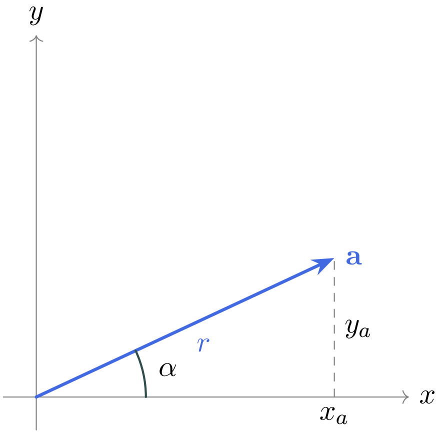
$$\begin{bmatrix} 1 & \tan\phi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 0 + \tan\phi \cdot 1 \\ 0\cdot 0 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} \tan\phi \\ 1 \end{bmatrix}$$

Upright axis tilts clockwise

Flat axis tilts up

$\tan 45^\circ = 1$: our shear

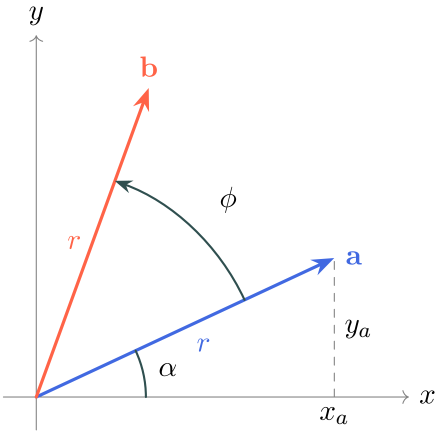
Rotation keeps the length, adds the angle



$$r = \sqrt{x_a^2 + y_a^2}$$
$$x_a = r\cos\alpha$$
$$y_a = r\sin\alpha$$

- Arrow $\mathbf{a}$ at angle α

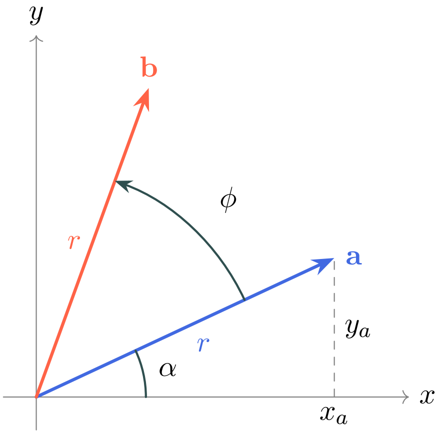
Rotation keeps the length, adds the angle



$$r = \sqrt{x_a^2 + y_a^2}$$
$$x_a = r\cos\alpha$$
$$y_a = r\sin\alpha$$

- Arrow $\mathbf{a}$ at angle α
- Turn by ϕ, counterclockwise

Rotation keeps the length, adds the angle



$$r = \sqrt{x_a^2 + y_a^2}$$

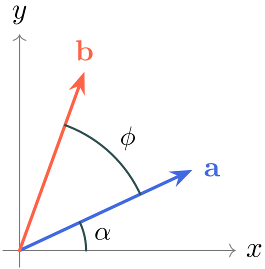
$$x_a = r\cos\alpha$$

$$y_a = r\sin\alpha$$

$\mathbf{b}$: length r, angle $\alpha + \phi$

- Arrow $\mathbf{a}$ at angle α
- Turn by ϕ, counterclockwise
- Same length, bigger angle

Expand with the angle-sum rules

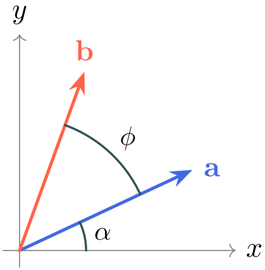


$$x_b = r\cos(\alpha + \phi)$$

angle $\alpha + \phi$

- Angle of $\mathbf{b}$: $\alpha + \phi$

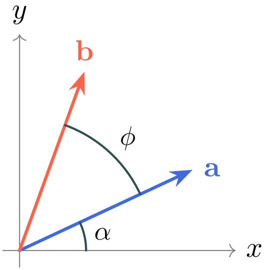
Expand with the angle-sum rules



$$x_b = r\cos(\alpha+\phi) \qquad \text{angle } \alpha+\phi$$
$$= r\cos\alpha\cos\phi - r\sin\alpha\sin\phi \qquad \text{cosine sum rule}$$

- Angle of $\mathbf{b}$: $\alpha+\phi$
- Split with the sum rule

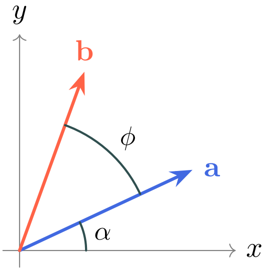
Expand with the angle-sum rules



$$
\begin{aligned}
x_b &= r\cos(\alpha+\phi) && \text{angle } \alpha+\phi \\
&= r\cos\alpha\cos\phi - r\sin\alpha\sin\phi && \text{cosine sum rule} \\
y_b &= r\sin(\alpha+\phi) && \text{angle } \alpha+\phi
\end{aligned}
$$

- Angle of $\mathbf{b}$: $\alpha+\phi$
- Split with the sum rule

Expand with the angle-sum rules



- Angle of $\mathbf{b}$: $\alpha + \phi$
- Split with the sum rule

$$
\begin{aligned}
x_b &= r\cos(\alpha + \phi) && \text{angle } \alpha + \phi \\
&= r\cos\alpha\cos\phi - r\sin\alpha\sin\phi && \text{cosine sum rule} \\
y_b &= r\sin(\alpha + \phi) && \text{angle } \alpha + \phi \\
&= r\sin\alpha\cos\phi + r\cos\alpha\sin\phi && \text{sine sum rule}
\end{aligned}
$$

Substitute, and the matrix appears

$$x_b = (r\cos\alpha)\cos\phi - (r\sin\alpha)\sin\phi$$ group r with α

Substitute, and the matrix appears

$$x_b = (r\cos\alpha)\cos\phi - (r\sin\alpha)\sin\phi \quad \text{group } r \text{ with } \alpha$$
$$= x_a\cos\phi - y_a\sin\phi \quad x_a = r\cos\alpha,\ y_a = r\sin\alpha$$

Substitute, and the matrix appears

$$\begin{aligned} x_b &= (r\cos\alpha)\cos\phi - (r\sin\alpha)\sin\phi && \text{group } r \text{ with } \alpha \\ &= x_a\cos\phi - y_a\sin\phi && x_a = r\cos\alpha,\ y_a = r\sin\alpha \end{aligned}$$

$$y_b = (r\sin\alpha)\cos\phi + (r\cos\alpha)\sin\phi \quad \text{group } r \text{ with } \alpha$$

Substitute, and the matrix appears

$$
\begin{aligned}
x_b &= (r\cos\alpha)\cos\phi - (r\sin\alpha)\sin\phi && \text{group } r \text{ with } \alpha \\
&= x_a\cos\phi - y_a\sin\phi && x_a = r\cos\alpha,\ \ y_a = r\sin\alpha
\end{aligned}
$$

$$
\begin{aligned}
y_b &= (r\sin\alpha)\cos\phi + (r\cos\alpha)\sin\phi && \text{group } r \text{ with } \alpha \\
&= y_a\cos\phi + x_a\sin\phi && \text{same substitution}
\end{aligned}
$$

Only x_a, y_a, ϕ remain

Substitute, and the matrix appears

$$
\begin{aligned}
x_b &= (r\cos\alpha)\cos\phi - (r\sin\alpha)\sin\phi && \text{group } r \text{ with } \alpha \\
&= x_a\cos\phi - y_a\sin\phi && x_a = r\cos\alpha,\ \ y_a = r\sin\alpha \\
y_b &= (r\sin\alpha)\cos\phi + (r\cos\alpha)\sin\phi && \text{group } r \text{ with } \alpha \\
&= y_a\cos\phi + x_a\sin\phi && \text{same substitution}
\end{aligned}
$$

$$
\begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix}
$$

Only x_a, y_a, ϕ remain

Read the rows off

Substitute, and the matrix appears

$$x_b = (r\cos\alpha)\cos\phi - (r\sin\alpha)\sin\phi \quad \text{group } r \text{ with } \alpha$$
$$= x_a\cos\phi - y_a\sin\phi \quad x_a = r\cos\alpha,\ \ y_a = r\sin\alpha$$

$$y_b = (r\sin\alpha)\cos\phi + (r\cos\alpha)\sin\phi \quad \text{group } r \text{ with } \alpha$$
$$= y_a\cos\phi + x_a\sin\phi \quad \text{same substitution}$$

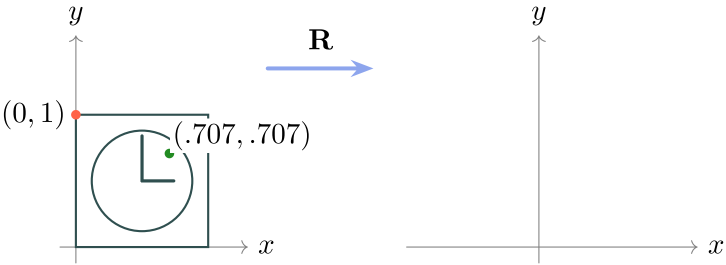
$$\begin{bmatrix} x_b \\ y_b \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} x_a \\ y_a \end{bmatrix}$$

$$\boxed{\text{rotate}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}}$$

Only x_a, y_a, ϕ remain

Read the rows off

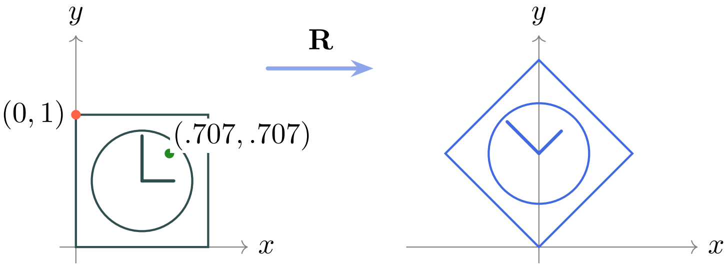
Rotate by $45°$: follow two points



$$\mathbf{R} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

- $45°$ counterclockwise

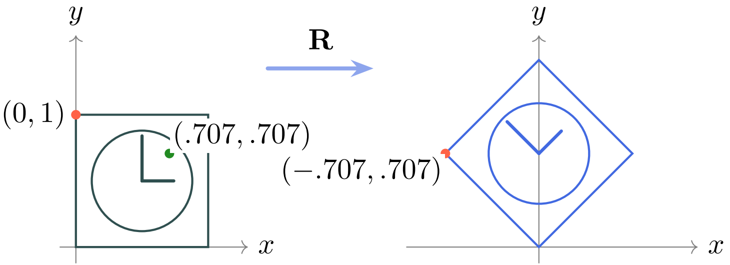
Rotate by $45°$: follow two points



$$\mathbf{R} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

- $45°$ counterclockwise

Rotate by $45°$: follow two points

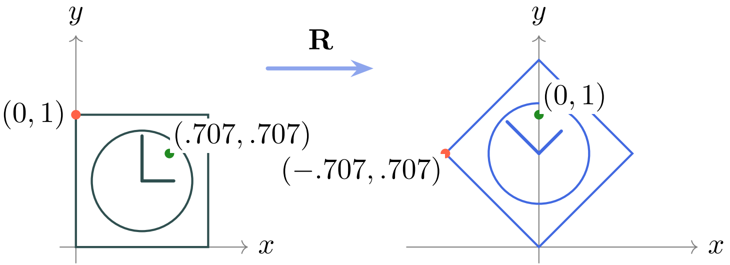


- $45°$ counterclockwise
- Corner $(0,1)$ swings left

$$\mathbf{R} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

$$\mathbf{R}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.707 \cdot 0 + (-0.707) \cdot 1 \\ 0.707 \cdot 0 + 0.707 \cdot 1 \end{bmatrix} = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}$$

Rotate by $45°$: follow two points



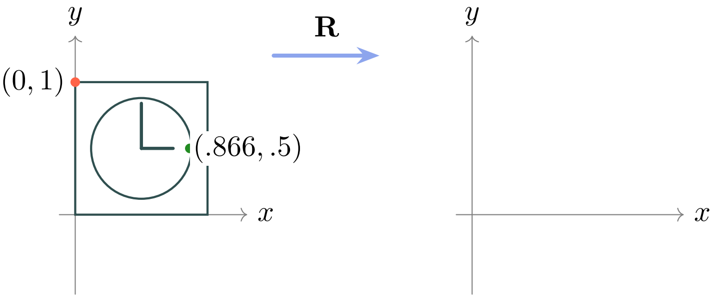
- $45°$ counterclockwise
- Corner $(0,1)$ swings left
- A face point reaches $(0,1)$

$$\mathbf{R} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix}$$

$$\mathbf{R}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.707 \cdot 0 + (-0.707) \cdot 1 \\ 0.707 \cdot 0 + 0.707 \cdot 1 \end{bmatrix} = \begin{bmatrix} -0.707 \\ 0.707 \end{bmatrix}$$

$$\mathbf{R}\begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix} = \begin{bmatrix} 0.707 \cdot 0.707 + (-0.707) \cdot 0.707 \\ 0.707 \cdot 0.707 + 0.707 \cdot 0.707 \end{bmatrix} \approx \begin{bmatrix} 0.5 - 0.5 \\ 0.5 + 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

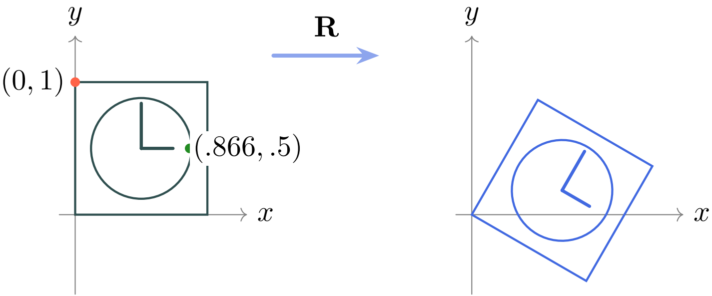
A negative angle turns clockwise



- Clockwise $30°$ is $\phi = -\frac{\pi}{6}$

$$\mathbf{R} = \begin{bmatrix} \cos\frac{-\pi}{6} & -\sin\frac{-\pi}{6} \\ \sin\frac{-\pi}{6} & \cos\frac{-\pi}{6} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix}$$

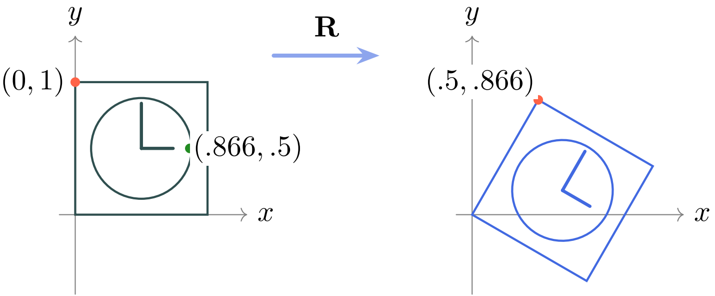
A negative angle turns clockwise



$$\mathbf{R} = \begin{bmatrix} \cos\frac{-\pi}{6} & -\sin\frac{-\pi}{6} \\ \sin\frac{-\pi}{6} & \cos\frac{-\pi}{6} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix}$$

- Clockwise 30° is $\phi = -\frac{\pi}{6}$
- Minus sign moves down

A negative angle turns clockwise

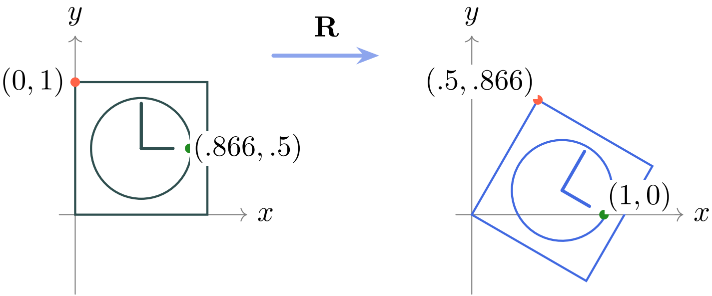


- Clockwise $30°$ is $\phi = -\frac{\pi}{6}$
- Minus sign moves down

$$\mathbf{R} = \begin{bmatrix} \cos\frac{-\pi}{6} & -\sin\frac{-\pi}{6} \\ \sin\frac{-\pi}{6} & \cos\frac{-\pi}{6} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix}$$

$$\mathbf{R}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 \cdot 0 + 0.5 \cdot 1 \\ -0.5 \cdot 0 + 0.866 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.866 \end{bmatrix}$$

A negative angle turns clockwise



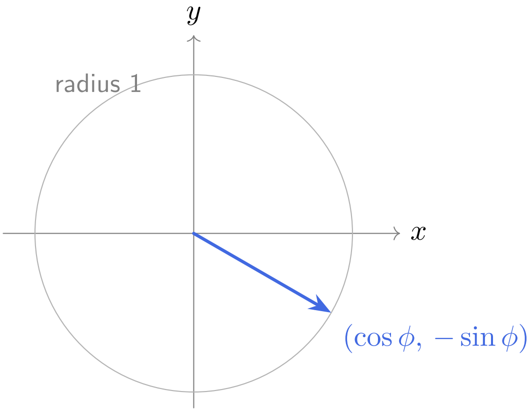
- Clockwise 30° is $\phi = -\frac{\pi}{6}$
- Minus sign moves down
- A face point lands on x

$$\mathbf{R} = \begin{bmatrix} \cos\frac{-\pi}{6} & -\sin\frac{-\pi}{6} \\ \sin\frac{-\pi}{6} & \cos\frac{-\pi}{6} \end{bmatrix} = \begin{bmatrix} 0.866 & 0.5 \\ -0.5 & 0.866 \end{bmatrix}$$

$$\mathbf{R}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.866 \cdot 0 + 0.5 \cdot 1 \\ -0.5 \cdot 0 + 0.866 \cdot 1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.866 \end{bmatrix}$$

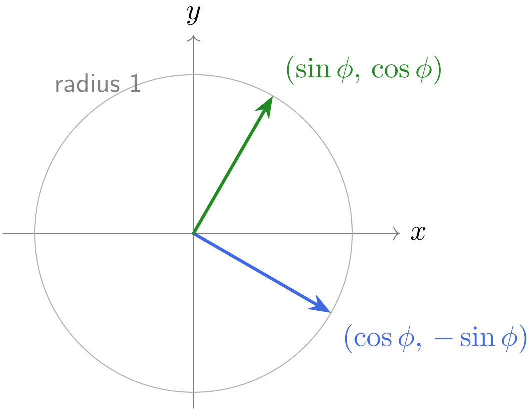
$$\mathbf{R}\begin{bmatrix} 0.866 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.866 \cdot 0.866 + 0.5 \cdot 0.5 \\ -0.5 \cdot 0.866 + 0.866 \cdot 0.5 \end{bmatrix} \approx \begin{bmatrix} 0.75 + 0.25 \\ -0.433 + 0.433 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Rotation rows: length one, at right angles



$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Rotation rows: length one, at right angles

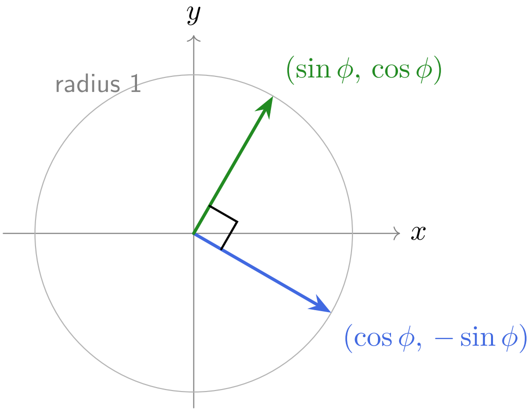


$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$\sin^2\phi + \cos^2\phi = 1$$

- Each row has length one

Rotation rows: length one, at right angles



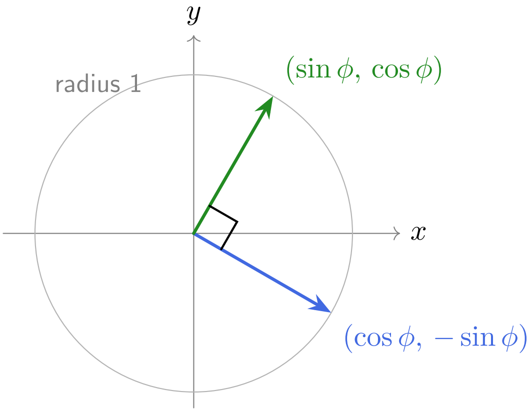
$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

$$\sin^2\phi + \cos^2\phi = 1$$

$$\cos\phi(-\sin\phi) + \sin\phi\cos\phi = 0$$

- Each row has length one
- Rows at right angles

Rotation rows: length one, at right angles



$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

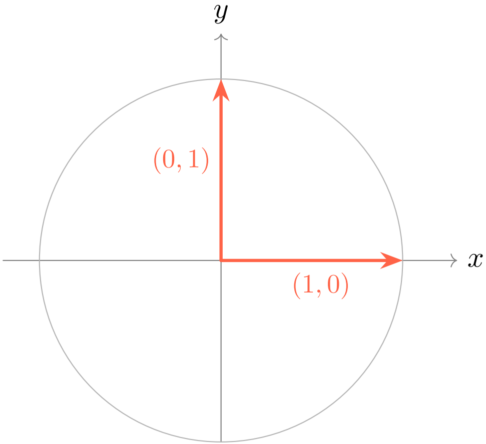
$$\sin^2\phi + \cos^2\phi = 1$$

$$\cos\phi(-\sin\phi) + \sin\phi\cos\phi = 0$$

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$$

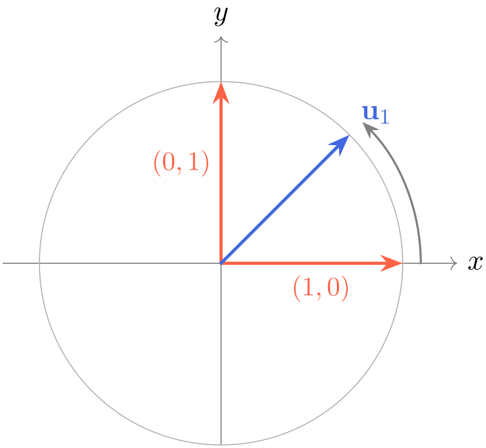
- Each row has length one
- Rows at right angles
- Rotations are orthogonal

Columns show where $\mathbf{x}$ and $\mathbf{y}$ land



$$\mathbf{R} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$$

Columns show where $\mathbf{x}$ and $\mathbf{y}$ land

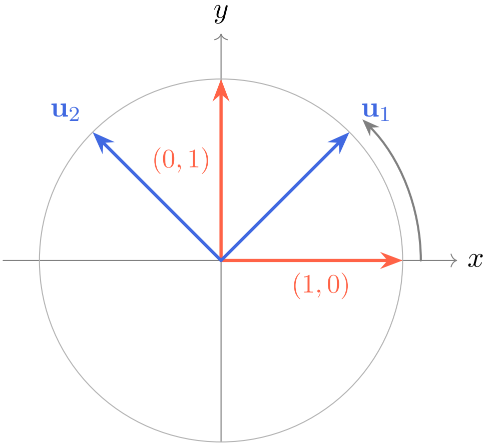


$$\mathbf{R} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$$

$$\mathbf{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\,\mathbf{u}_1 + 0\,\mathbf{u}_2 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

- Columns: where $\mathbf{x}$, $\mathbf{y}$ land

Columns show where $\mathbf{x}$ and $\mathbf{y}$ land

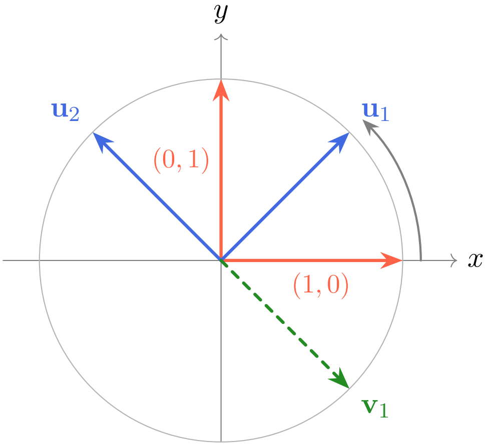


$$\mathbf{R} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$$

$$\mathbf{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\,\mathbf{u}_1 + 0\,\mathbf{u}_2 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

- Columns: where $\mathbf{x}$, $\mathbf{y}$ land

Columns show where $\mathbf{x}$ and $\mathbf{y}$ land



$$\mathbf{R} = \begin{bmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{bmatrix} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 \end{bmatrix}$$

$$\mathbf{R} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1\,\mathbf{u}_1 + 0\,\mathbf{u}_2 = \begin{bmatrix} 0.707 \\ 0.707 \end{bmatrix}$$

$$\mathbf{R} \begin{bmatrix} 0.707 \\ -0.707 \end{bmatrix} \approx \begin{bmatrix} 0.5 + 0.5 \\ 0.5 - 0.5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

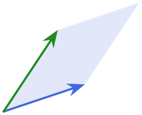
- Columns: where $\mathbf{x}$, $\mathbf{y}$ land
- Rows: what lands on them
- Read the picture from the numbers

Recap: the rotation matrix

$$\text{rotate}(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

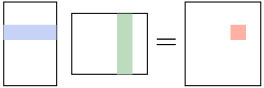
Summary

$$|\mathbf{ab}| = x_a y_b - y_a x_b$$



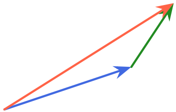
Signed area of two arrows

$$p_{ij} = a_{i1}b_{1j} + \cdots + a_{im}b_{mj}$$



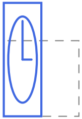
Row meets column

$$\mathbf{y} = x_1\mathbf{c}_1 + x_2\mathbf{c}_2$$



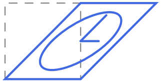
Output mixes the columns

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$$



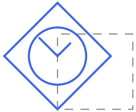
Scale: stretch along the axes

$$\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$$



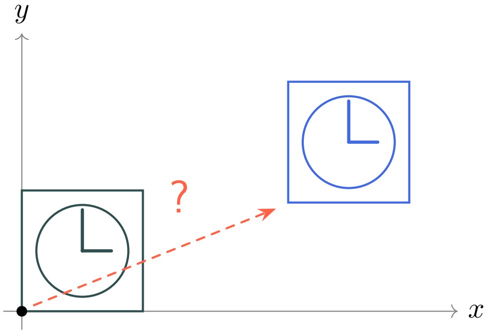
Shear: push sideways

$$\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$



Rotate: turn counterclockwise

Next lecture: moving the clock off the origin



Shirley §7.1.5, §7.3, §6.3–6.3.1, §7.4

- Slide the clock: a new trick
- Chain moves: order matters
- Undo a chain of moves