Finger Exercises 13

The Cross Product and the Camera

≈ 55 minutes, plus a bonus
CS 453 — Computer Graphics

You need a pencil, a ruler and your right hand. Every number was chosen to come out by hand, so no calculator is needed. Conventions are the lecture's: x right, y up, z toward you; bold letters are points and arrows, plain letters are numbers, and x_a, y_a, z_a are the three numbers of $\mathbf{a}$. The right hand: thumb on the first arrow, index finger on the second, and the middle finger gives the cross product. Nothing here is an assignment task; the sheet is Lecture 13 only.

Short on time? Do Exercises 2, 7, 9 and 10 (about 24 minutes): the slot drill, cameras to order, pixel centres and the three legs of a ray.

Part 1 · The cross product

Exercises 1–4

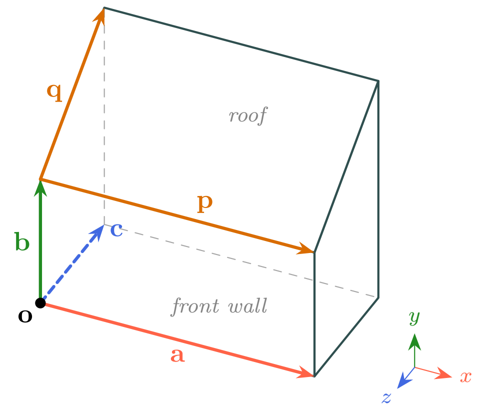
A lean-to shed stands on the floor $y = 0$. It is 4 wide along x and 2 deep along z. Its front wall is 2 tall and its back wall 3.5 tall, so the roof slopes down toward you. Its front-left-bottom corner is the origin $\mathbf{o}$.

1 Right hand on a shed

about 4 min

Read a cross product off a picture before any formula: the direction from three fingers, the length from an area.

From $\mathbf{o}$ run three edges: $\mathbf{a} = (4, 0, 0)$ along the front, $\mathbf{b} = (0, 2, 0)$ up the corner, and $\mathbf{c} = (0, 0, -2)$ back along the left side (dashed, because the shed hides it).



product	points along	length: the area of	numbers
$\mathbf{a} \times \mathbf{b}$	$-\mathbf{c}$, toward you	8: the front wall, 4 by 2	(0, 0, 8)
$\mathbf{b} \times \mathbf{a}$			
$\mathbf{b} \times \mathbf{c}$			
$\mathbf{c} \times \mathbf{b}$			
$\mathbf{c} \times \mathbf{a}$			
$\mathbf{a} \times \mathbf{c}$			
$\mathbf{c} \times \mathbf{c}$			

(a) Fill the table without the slot formula. The right hand gives the direction; the length is the area of the rectangle that the two arrows span.

(b) Three of the products point *out of* the shed, through a wall or through the floor. Which three? Look at the order of the two arrows in each: what pattern do you see?

(c) The roof. From its front-left corner $(0, 2, 0)$ run $\mathbf{p} = (4, 0, 0)$ along the front edge and $\mathbf{q} = (0,\ 1.5,\ -2)$ up the slope. Check with a dot product that $\mathbf{p}$ and $\mathbf{q}$ are perpendicular, and find the roof's area. Then predict $\mathbf{p} \times \mathbf{q}$: which way does it point, and how long is it? (Exercise 2 computes it.)

2 The slot drill

about 7 min

The slot formula until it is automatic, with two checks that cost one dot product each.

$\mathbf{a} \times \mathbf{b} = \big(y_a z_b - z_a y_b,\ \ z_a x_b - x_a z_b,\ \ x_a y_b - y_a x_b\big)$. Each slot skips its own letter and uses the next two on the wheel $x \to y \to z \to x$: wheel order first, swapped order second. **Checks:** the answer is perpendicular to both arrows, so $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} = 0$ and $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$; its length is the area of the parallelogram. The first row is done for you.

	$\mathbf{a}$	$\mathbf{b}$	x slot	y slot	z slot	$\cdot\,\mathbf{a}$	$\cdot\,\mathbf{b}$	length
	(2, 2, 1)	(1, 0, 1)	2	−1	−2	0	0	3
1	(2, 3, 0)	(4, 1, 0)	____	____	____	____	____	____
2	(−3, 1, 0)	(3, −2, 2)	____	____	____	____	____	____
3	(−3, 0, 1)	(3, 2, 2)	____	____	____	____	____	____
4	(2, −1, 3)	(−4, 2, −6)	____	____	____	____	____	____
5	$\mathbf{p} = (4, 0, 0)$	$\mathbf{q} = (0, 1.5, -2)$	____	____	____	____	____	____

(a) Row 1's arrows lie on the wall $z = 0$. Which slots vanish, and why? The slot that survives is lecture 4's 2D cross product: what does its sign say about where $\mathbf{b}$ lies compared with $\mathbf{a}$?

(b) Row 4 gives the zero vector. What do its two arrows have in common?

(c) Row 5 is the roof. Does it match your prediction from Exercise 1(c)?

3 Which check catches which slip?

about 3 min

Checking an answer without redoing it, and the one slip that no dot product can catch.

Four students hand in $\mathbf{a} \times \mathbf{b}$. Test each claim with the two dot products only; do not redo any slot yet.

student	$\mathbf{a}$	$\mathbf{b}$	claimed $\mathbf{a} \times \mathbf{b}$	claim $\cdot\,\mathbf{a}$	claim $\cdot\,\mathbf{b}$	passes both?
A	(3, 0, 1)	(1, 2, 0)	(−2, 1, 6)	____	____	____
B	(2, 1, 3)	(1, 0, −1)	(−1, −5, −1)	____	____	____
C	(2, 0, −1)	(3, 1, 2)	(−1, −7, 2)	____	____	____
D	(1, −1, 2)	(0, 2, 1)	(5, 1, −2)	____	____	____

(a) Which claims fail a check? Does a claim have to fail both checks to be wrong?

(b) A and D pass both checks. Recompute the z slot of each. Which one is wrong anyway? Why can no dot-product check ever catch this slip?

(c) Name the slip in each wrong claim: the alphabet order in the y slot $(x_a z_b - z_a x_b)$, a lost double minus, or the two arrows taken in the wrong order.

4 What if . . . ?

about 4 min

Predict a cross product from the lecture's rules: order, stretching, adding, and an arrow crossed with itself.

Start from row 2 of Exercise 2: $\mathbf{a} = (-3,\ 1,\ 0)$, $\mathbf{b} = (3,\ -2,\ 2)$ and $\mathbf{a} \times \mathbf{b} = (2,\ 6,\ 3)$. Predict each result without the slot formula, and name the rule you used.

	what is asked	prediction	rule
1	$\mathbf{b} \times \mathbf{a}$	______	______
2	$(2\mathbf{a}) \times \mathbf{b}$	______	______
3	$\mathbf{a} \times (-\mathbf{b})$	______	______
4	$\mathbf{a} \times (\mathbf{b} + 3\mathbf{a})$	______	______
5	$(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} - \mathbf{b})$	______	______
6	$\mathbf{a} \times \mathbf{b}$ after both arrows are moved to start at $(5,\ 5,\ 5)$	______	______
7	the area of the triangle whose two sides are $\mathbf{a}$ and $\mathbf{b}$	______	______

(a) Check row 5 with the slot formula.

(b) Row 4 in a picture: the parallelogram of $\mathbf{a}$ and $\mathbf{b} + 3\mathbf{a}$ has the same base as the parallelogram of $\mathbf{a}$ and $\mathbf{b}$. Why does it also have the same height?

Part 2 · Frames from arrows

Exercises 5–7

A basis for 3D is three arrows $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, each one unit long and each pair at right angles (orthonormal). In this course it is also right-handed: $\mathbf{w} = \mathbf{u} \times \mathbf{v}$.

5 Sort the bases

about 3 min

The three tests every frame must pass: lengths, right angles, handedness.

	$\mathbf{u}$	$\mathbf{v}$	$\mathbf{w}$	lengths all 1?	$\mathbf{u}\cdot\mathbf{v},\ \mathbf{v}\cdot\mathbf{w},\ \mathbf{w}\cdot\mathbf{u}$	$\mathbf{u} \times \mathbf{v}$	verdict
	$(0, 0, 1)$	$(1, 0, 0)$	$(0, 1, 0)$	yes	$0,\ 0,\ 0$	$(0, 1, 0) = \mathbf{w}$	right-handed
1	$(0.6,\ 0,\ -0.8)$	$(0,\ 1,\ 0)$	$(0.8,\ 0,\ 0.6)$	______	______	______	______
2	$\frac{1}{3}(1,\ 2,\ 2)$	$\frac{1}{3}(2,\ 1,\ -2)$	$\frac{1}{3}(2,\ -2,\ 1)$	______	______	______	______
3	$(1,\ 1,\ 0)$	$(-1,\ 1,\ 0)$	$(0,\ 0,\ 2)$	______	______	______	______
4	$(0.6,\ 0.8,\ 0)$	$(0,\ 0.6,\ 0.8)$	$(0.8,\ -0.6,\ 0)$	______	______	______	______

(a) Fill the table. Each verdict is one of: right-handed orthonormal, left-handed orthonormal, or not orthonormal.

(b) Repair rows 2 and 3 with the smallest change you can.

(c) For row 1, also compute $\mathbf{v} \times \mathbf{w}$ and $\mathbf{w} \times \mathbf{u}$. Compare them with the wheel of the axes, $\mathbf{y} \times \mathbf{z} = \mathbf{x}$ and $\mathbf{z} \times \mathbf{x} = \mathbf{y}$.

6 One arrow, many frames

about 5 min

A helper arrow completes a frame around one given arrow; another helper spins the frame, and some helpers fail.

Build frames around the roof of Exercise 1. Its cross product from Exercise 2, cut to length one, is $\mathbf{w} = (0,\ 0.8,\ 0.6)$: it stands perpendicular to the roof. The lecture's recipe with a helper $\mathbf{t}$: $\mathbf{u} = (\mathbf{t} \times \mathbf{w})/\|\mathbf{t} \times \mathbf{w}\|$, then $\mathbf{v} = \mathbf{w} \times \mathbf{u}$.

(a) The book's helper: copy $\mathbf{w}$ and replace its number of smallest size by 1, which gives $\mathbf{t} = (1,\ 0.8,\ 0.6)$. Build $\mathbf{u}$ and $\mathbf{v}$. Where do they point on the roof? (Use the picture of Exercise 1.) Check that $\mathbf{u} \times \mathbf{v} = \mathbf{w}$.

(b) Build the frame again with the helper $\mathbf{t} = (0, 1, 0)$. How is the new pair $\mathbf{u}, \mathbf{v}$ related to the pair from (a)?

(c) The helper $(1, 0, 0)$ gives exactly the frame of (a). Why? Hint: subtract $(1, 0, 0)$ from the book's helper.

(d) Which of these helpers fail: $(0,\ -4,\ -3)$, $(0,\ 1.6,\ 1.2)$, $(0,\ 0,\ 1)$? The helper $(0,\ 0.8,\ 0.61)$ does not fail: how long is $\mathbf{t} \times \mathbf{w}$, and why is it still a poor choice?

7 Cameras to order

about 8 min

The camera recipe on five cameras: the view direction decides $\mathbf{w}$*, up decides the rest, and one camera breaks it.*

Inputs: a view direction and an up arrow. Then $\mathbf{a} = -(\text{view direction})$ and $\mathbf{b} = \text{up}$; $\mathbf{w} = \mathbf{a}/\|\mathbf{a}\|$, $\mathbf{u} = (\mathbf{b} \times \mathbf{w})/\|\mathbf{b} \times \mathbf{w}\|$, $\mathbf{v} = \mathbf{w} \times \mathbf{u}$. Check every row: $\mathbf{u} \times \mathbf{v}$ must give $\mathbf{w}$. The first row is done for you.

	camera	view direction	up	$\mathbf{w}$	$\mathbf{u}$	$\mathbf{v}$
	level, looking along $-z$	$(0, 0, -3)$	$(0, 1, 0)$	$(0, 0, 1)$	$(1, 0, 0)$	$(0, 1, 0)$
1	level, turned	$(-3,\ 0,\ -4)$	$(0, 1, 0)$	______	______	______
2	a drone at $(12,\ 17,\ 16)$ looking at the point $(0,\ 2,\ 0)$	______	$(0, 1, 0)$	______	______	______
3	looking straight down	$(0,\ -2,\ 0)$	$(0, 1, 0)$	______	______	______
4	looking straight down	$(0,\ -2,\ 0)$	$(0,\ 0,\ -1)$	______	______	______

(a) In which rows is $\mathbf{v}$ the same as up? What do those cameras have in common?

(b) With up $= (0, 1, 0)$, every $\mathbf{u}$ has $y = 0$. Why? What does that mean for the picture's bottom edge? Rows 1 and 2 even share their $\mathbf{u}$: why?

(c) Row 2: $\mathbf{v}$ is up, straightened. Take away from up its part along $\mathbf{w}$, $\text{up} - (\text{up} \cdot \mathbf{w})\,\mathbf{w}$ (lecture 4's "along" and "across"), and cut the rest to length one. Do you get row 2's $\mathbf{v}$? How many degrees does $\mathbf{v}$ lean away from up, and how far below the horizon does the drone look?

(d) Row 3 breaks. At which step, and why? In row 4, which way is the top of the picture?

(e) Many programs call the camera's arrows forward, right and up, with forward $\mathbf{f} = -\mathbf{w}$. They compute $\text{right} = (\mathbf{f} \times \text{up})/\|\mathbf{f} \times \text{up}\|$ and then $\text{up}' = \text{right} \times \mathbf{f}$. Use the rules of Exercise 4 to show that right $= \mathbf{u}$ and $\text{up}' = \mathbf{v}$.

Part 3 · One ray per pixel

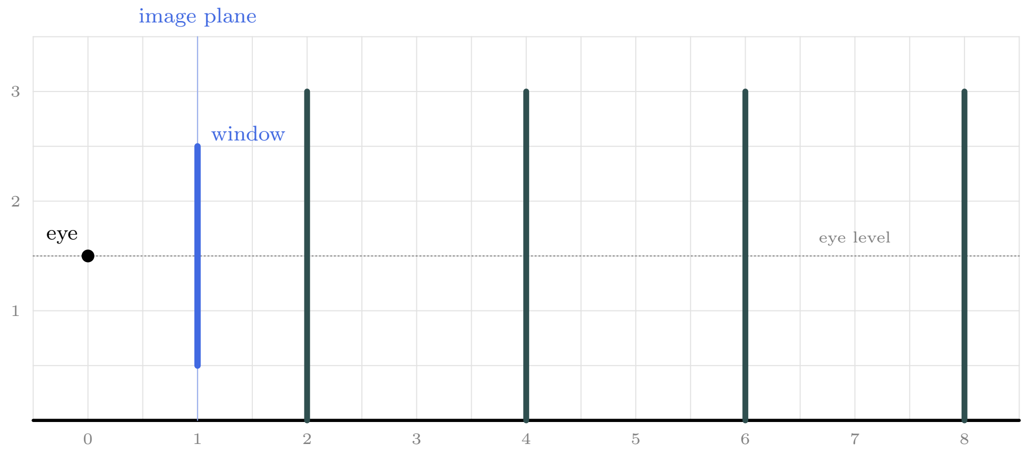
Exercises 8–11

8 Perspective with a ruler

about 6 min

Similar triangles by drawing: far things look small, and a longer d zooms in.

Seen from the side: an eye 1.5 above the ground, an image plane $d = 1$ in front of it, and four fence posts, each 3 tall, at distances 2, 4, 6 and 8 from the eye. The window is the part of the image plane from 1 below the eye to 1 above it.



(a) Draw the lines from the top and from the foot of each post to the eye. Mark where each line crosses the image plane, and measure each post's image.

(b) Similar triangles give: image height = post height × d / distance. Check your four measurements.

(c) The posts are equally spaced. Are the feet of their images equally spaced? Where would the foot of a post very far away be drawn?

(d) Where does the dotted eye level cut each post's image, and why there?

(e) Parallel projection instead: every point slides straight across, level, to the image plane. How tall is each image now? How many posts can you see?

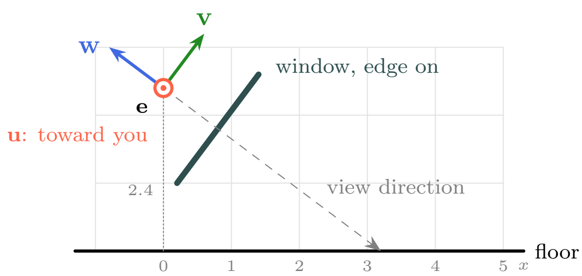
(f) Zoom: move the image plane to $d = 1.5$, and keep the window 2 tall around eye level. Which posts still fit completely inside the window?

Camera S

Exercises 9–11 and the bonus

The eye is $\mathbf{e} = (0,\ 2.4,\ 0)$, 2.4 above the floor $y = 0$. The camera looks along $(4, -3, 0)$, down and along x, with up $(0, 1, 0)$. The recipe of Exercise 7 gives $\mathbf{u} = (0,\ 0,\ 1)$, $\mathbf{v} = (0.6,\ 0.8,\ 0)$ and $\mathbf{w} = (-0.8,\ 0.6,\ 0)$.

The window hangs $d = 1$ in front of the eye, with $l = -1.5$, $r = 1.5$, $b = -1$, $t = 1$, and is cut into $n_x = 6$ by $n_y = 4$ pixels. Check $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ before you start.

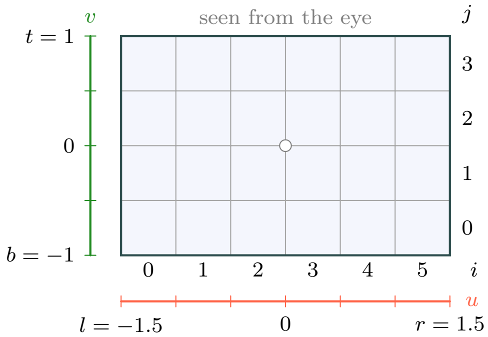


seen from the side: x right, y up, z toward you

9 Pixel centres, both ways

about 4 min

Pixel coordinates (i, j) to camera coordinates (u, v) and back, and what the half cell is for.



Camera S's window. The small circle is straight ahead of the eye. Columns i count from the left and rows j from the bottom, both from 0, and a pixel's ray passes through the centre of its cell:

$$u = l + (r - l)\,\frac{i + 0.5}{n_x}, \qquad v = b + (t - b)\,\frac{j + 0.5}{n_y}.$$

pixel (i, j)	$(0, 0)$	$(5, 3)$	$(2, 1)$	$(4, 0)$	____	____
camera coordinates (u, v)	$(-1.25,\ -0.75)$	____	____	____	$(0.25,\ 0.75)$	$(-0.75,\ -0.25)$

(a) Fill the table, and mark each centre on the picture.

(b) Which pixel contains the spot $(u, v) = (1.1,\ -0.4)$? (It is not a centre.)

(c) Which pixel's centre is straight ahead of the eye, at $(0, 0)$?

(d) A program forgot the 0.5. Through which spot does pixel $(0, 0)$'s ray pass now? What happens to the whole picture?

(e) An image file stores the top row first, as its row 0. In which row of the file does pixel $(2, 1)$ go?

(f) Someone asks for 6×3 pixels but keeps this window. How wide and how tall is a cell now? Why would a round ball come out squashed on a screen of square pixels?

10 Three legs

about 5 min

The ray through a pixel: forward d, across u, up v, added slot by slot.

For Camera S every ray starts at $\mathbf{e}$, and direction $= -d\,\mathbf{w} + u\,\mathbf{u} + v\,\mathbf{v}$.

pixel	(u, v)	$-d\,\mathbf{w}$	$u\,\mathbf{u}$	$v\,\mathbf{v}$	direction
$(4, 2)$	$(0.75,\ 0.25)$	$(0.8,\ -0.6,\ 0)$	$(0,\ 0,\ 0.75)$	$(0.15,\ 0.2,\ 0)$	$(0.95,\ -0.4,\ 0.75)$
$(0, 0)$	____	____	____	____	____
$(5, 1)$	____	____	____	____	____
$(2, 3)$	____	____	____	____	____

(a) Fill the table.

(b) Check pixel $(0, 0)$'s ray: at $t = 1$ it reaches $\mathbf{s} = \mathbf{e} +$ direction, which must have camera coordinates $(u, v, -d)$. Find them with lecture 9's recipe: subtract $\mathbf{e}$, then dot with $\mathbf{u}$, with $\mathbf{v}$ and with $\mathbf{w}$.

(c) Pixel $(2, 3)$'s direction has $y = 0$. Which other pixels share this? What do their rays look at?

(d) A student wrote $+d\,\mathbf{w}$. Which way does their ray for pixel $(4, 2)$ go?

(e) An orthographic Camera S (same frame and window, but every ray starts on its pixel and points along $-\mathbf{w}$): give the origin and the direction of pixel $(4, 2)$'s ray.

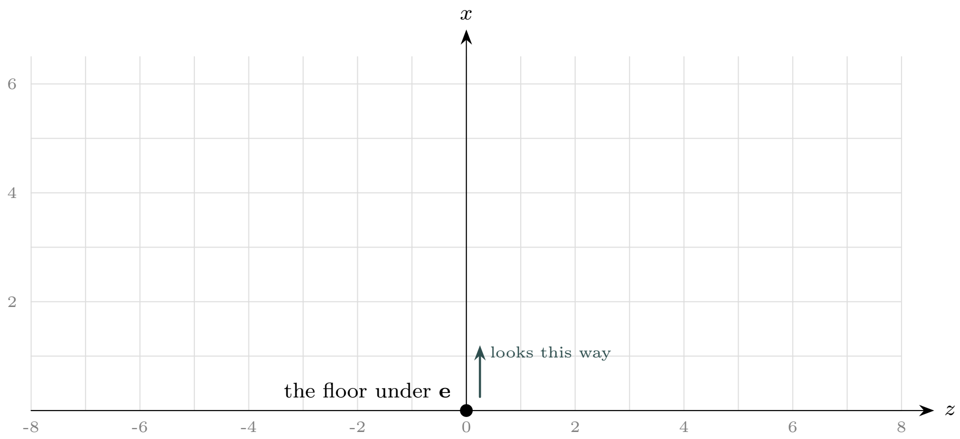
11 Footprints on the floor

about 7 min

Where each row of pixels meets the floor: perspective, seen from above.

A ray from Camera S meets the floor $y = 0$ when $2.4 + t \times$ (the direction's y) $= 0$. Along one row of pixels v stays the same, so the direction's x and y are the same in every column; only its z, which is u, changes.

row j	v	direction's x	direction's y	t at the floor	lands at $x =$
0	______	______	______	______	______
1	______	______	______	______	______
2	______	______	______	______	______
3	______	______	______	______	______



the floor seen from above; one square per unit

(a) Fill the table.

(b) In a row, the pixel with camera coordinate u lands at $z = t\,u$. Plot the six landing points of rows 0, 1 and 2, and join each row. What shape do they make? Why are the far rows wider and farther apart?

(c) Row 3 never lands. What does that row of the picture show?

(d) The floor is tiled in 1 by 1 tiles. How far apart do the rays of two neighbouring pixels land in row 0, and in row 2? What goes wrong in the picture of a finely tiled floor far away?

(e) Orthographic Camera S: every ray points along $-\mathbf{w}$ and starts at $\mathbf{e} + u\,\mathbf{u} + v\,\mathbf{v}$. Where do the rays of the four corner pixels land? Draw that footprint too. What changed, and does this camera see a horizon?

Try it tonight optional, about 10 minutes, no marks

You need a card, scissors, a ruler, and a window pane with a whiteboard marker (or a clear plastic folder).

1. Cut a hole 3 cm wide and 2 cm tall in the card: the shape of Camera S's window. Close one eye and hold the card 10 cm in front of the other, then 20 cm. Which shows more of the room? Which one is the longer d?
2. Stand 3 m from a wall. How wide a strip of the wall should the hole show at 10 cm, and at 20 cm? Check it against something on the wall.
3. Alberti's window. Rest your head against something still, close one eye, and trace the outline of a door or a car far away on the pane. Measure the traced height h and the distance d from your eye to the glass, and pace out the distance D to the object. Its real height should be about $h\,D/d$.
4. Make a camera frame with your right hand: thumb to your right ($\mathbf{u}$), index finger up ($\mathbf{v}$). The middle finger points back at your face: that is $\mathbf{w}$, and you look along $-\mathbf{w}$. Now look down at the floor, keeping your head upright. Which finger stays level?

12 Bonus: from a point to a pixel ★

about 10 min

For the brave. The lecture went from a pixel to a ray. Run it backwards: given a point of the scene, find the pixel that shows it. Camera S throughout.

(a) **Camera coordinates.** A small lantern hangs at $\mathbf{P} = (3.8,\ 0.8,\ 3)$. Use lecture 9's recipe: subtract $\mathbf{e}$, then dot with $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ to get u_P, v_P, w_P. The camera looks along $-\mathbf{w}$, so a point in front of it has $w_P < 0$. Is the lantern in front?

(b) **Onto the window.** The window lies d in front of the eye, and the lantern lies $-w_P$ in front of it. By similar triangles, the line from $\mathbf{e}$ to $\mathbf{P}$ crosses the window at $(u, v) = (u_P,\ v_P) \times d/(-w_P)$. Find this spot, then the pixel that contains it. At which t does that pixel's ray, from Exercise 10, pass through $\mathbf{P}$?

(c) Why is the scale $d/(-w_P)$ and not d divided by the lantern's distance from the eye?

(d) A flag flies at $(4.8,\ 3.8,\ -1.5)$. Is it in the picture?

(e) A bird sits at $(-1.3,\ 4,\ 1.5)$. A program that skips the test of (a) draws the bird anyway. In which pixel, and why is that wrong?

(f) Push the arrow $(0, -1, 0)$, straight down, through the same steps: its camera coordinates, then $\times\ d/(-w)$. Every vertical edge in the scene, a post or a door frame, points toward this spot in the picture. Is the spot inside the window? What happens to the vertical edges in a picture taken by a camera tilted down, and in one taken by a level camera?

Answer Key

Check your work *after* attempting everything. If an answer surprises you, that item is worth re-reading in the slides.

1.

product	points along	length: the area of	numbers
$\mathbf{a}\times\mathbf{b}$	$-\mathbf{c}$, toward you	8: the front wall, 4 by 2	$(0,0,8)$
$\mathbf{b}\times\mathbf{a}$	$+\mathbf{c}$, away from you	8: the front wall	$(0,0,-8)$
$\mathbf{b}\times\mathbf{c}$	$-\mathbf{a}$, to the left	4: the lower 2 by 2 square of the left wall	$(-4,0,0)$
$\mathbf{c}\times\mathbf{b}$	$+\mathbf{a}$, to the right	4: the same square	$(4,0,0)$
$\mathbf{c}\times\mathbf{a}$	$-\mathbf{b}$, down	8: the floor, 2 by 4	$(0,-8,0)$
$\mathbf{a}\times\mathbf{c}$	$+\mathbf{b}$, up	8: the floor	$(0,8,0)$
$\mathbf{c}\times\mathbf{c}$	nowhere	0: no area	$(0,0,0)$

1b. $\mathbf{a}\times\mathbf{b}$ (out through the front wall), $\mathbf{b}\times\mathbf{c}$ (through the left wall) and $\mathbf{c}\times\mathbf{a}$ (through the floor). Their arrows go round one cycle, $\mathbf{a}\to\mathbf{b}\to\mathbf{c}\to\mathbf{a}$, like the wheel $x\to y\to z\to x$. The other order points into the shed.

1c. $\mathbf{p}\cdot\mathbf{q}=0$, so the roof is a rectangle, 4 by $\|\mathbf{q}\|=\sqrt{1.5^2+2^2}=2.5$: area 10. Thumb to the right along $\mathbf{p}$, index finger up the slope along $\mathbf{q}$: the middle finger points up and toward you, perpendicular to the roof. Prediction: 10 long, up and toward you, out of the roof.

2.

	x slot	y slot	z slot	$\mathbf{a}\times\mathbf{b}$	length
1	$3\cdot0-0\cdot1=0$	$0\cdot4-2\cdot0=0$	$2\cdot1-3\cdot4=-10$	$(0,0,-10)$	10
2	$1\cdot2-0\,(-2)=2$	$0\cdot3-(-3)\cdot2=6$	$(-3)(-2)-1\cdot3=3$	$(2,6,3)$	7
3	$0\cdot2-1\cdot2=-2$	$1\cdot3-(-3)\cdot2=9$	$(-3)\cdot2-0\cdot3=-6$	$(-2,9,-6)$	11
4	$(-1)(-6)-3\cdot2=0$	$3\,(-4)-2\,(-6)=0$	$2\cdot2-(-1)(-4)=0$	$(0,0,0)$	0
5	$0\,(-2)-0\,(1.5)=0$	$0\cdot0-4\,(-2)=8$	$4\,(1.5)-0\cdot0=6$	$(0,8,6)$	10

Both dot checks give 0 in every row.

2a. The x and y slots vanish: each of their products contains a z number, and both z numbers are 0. The z slot, $x_ay_b-y_ax_b=-10$, is lecture 4's $a_xb_y-a_yb_x$. It is negative, so $\mathbf{b}$ lies to the right of $\mathbf{a}$ (clockwise from it), and the cross product points away from you.

2b. $\mathbf{b}=-2\mathbf{a}$: the two arrows lie on one line and span no area.

2c. Yes: $(0,8,6)$ points up ($y=8$) and toward you ($z=6$), and its length $\sqrt{64+36}=10$ is the roof's area.

3.

student	claim $\cdot\,\mathbf{a}$	claim $\cdot\,\mathbf{b}$	passes both?
A	$-6+0+6=0$	$-2+2+0=0$	yes
B	$-2-5-3=-10$	$-1+0+1=0$	no
C	$-2+0-2=-4$	$-3-7+4=-6$	no
D	$5-1-4=0$	$0+2-2=0$	yes

3a. B and C. One failed check is enough: B passes the check with $\mathbf{b}$ and is still wrong.

3b. A's z slot is $3\cdot2-0\cdot1=6$, as claimed. D's is $1\cdot2-(-1)\cdot0=2$, but D claims -2: D is wrong. D handed in $\mathbf{b}\times\mathbf{a}=-(\mathbf{a}\times\mathbf{b})$. The reversed arrow is just as perpendicular to both inputs and just as long, so no dot product and no length check can see the flip. Only the right hand, or one recomputed slot, can.

3c. B: alphabet order in the y slot, $x_az_b-z_ax_b=-2-3=-5$ instead of $z_ax_b-x_az_b=3+2=5$. C: a lost double minus in the x slot, $0\cdot2-(-1)\cdot1=+1$, not -1. D: the arrows in the wrong order. (True answers: B $(-1,5,-1)$, C $(1,-7,2)$, D $(-5,-1,2)$.)

4.

	prediction	rule
1	$(-2,-6,-3)$	swap the order, flip the arrow
2	$(4,12,6)$	a number moves to the front (stretch)
3	$(-2,-6,-3)$	stretch with $k=-1$
4	$(2,6,3)$	sum rule, and $\mathbf{a}\times\mathbf{a}=\mathbf{0}$: $\mathbf{a}\times\mathbf{b}+3\,\mathbf{a}\times\mathbf{a}$
5	$(-4,-12,-6)$	multiply out: $\mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}=-2\,(\mathbf{a}\times\mathbf{b})$
6	$(2,6,3)$	an arrow is a step, not a place: moving it changes nothing
7	3.5	half the parallelogram, $\|\mathbf{a}\times\mathbf{b}\|=7$

4a. $\mathbf{a}+\mathbf{b}=(0,-1,2)$, $\mathbf{a}-\mathbf{b}=(-6,3,-2)$: slots $(-1)(-2)-2\cdot3=-4$, $2(-6)-0(-2)=-12$, $0\cdot3-(-1)(-6)=-6$. It agrees.

4b. Adding $3\mathbf{a}$ slides the tip of $\mathbf{b}$ parallel to the base $\mathbf{a}$, so its distance from the line of $\mathbf{a}$, the height, does not change: a shear, which keeps the area (lecture 7). The two parallelograms also lie on the same plane, so the direction is the same too.

5.

	lengths all 1?	$\mathbf{u}\cdot\mathbf{v}$, $\mathbf{v}\cdot\mathbf{w}$, $\mathbf{w}\cdot\mathbf{u}$	$\mathbf{u}\times\mathbf{v}$	verdict
1	yes	0, 0, 0	$(0.8, 0, 0.6) = \mathbf{w}$	right-handed orthonormal
2	yes: $\sqrt{1+4+4}/3$	0, 0, 0	$\frac{1}{9}(-6, 6, -3) = -\mathbf{w}$	left-handed orthonormal
3	no: $\sqrt{2}$, $\sqrt{2}$, 2	0, 0, 0	$(0, 0, 2) = \mathbf{w}$	not orthonormal: lengths
4	yes	0.48, -0.36, 0	$(0.64, -0.48, 0.36)$	not orthonormal: angles

5b. Row 2: flip $\mathbf{w}$ to $\frac{1}{3}(-2, 2, -1)$. Row 3: divide each arrow by its length: $\frac{1}{\sqrt{2}}(1, 1, 0)$, $\frac{1}{\sqrt{2}}(-1, 1, 0)$, $(0, 0, 1)$.

5c. $\mathbf{v}\times\mathbf{w} = (0.6, 0, -0.8) = \mathbf{u}$ and $\mathbf{w}\times\mathbf{u} = (0, 1, 0) = \mathbf{v}$. A right-handed orthonormal basis obeys the same wheel as the axes, $\mathbf{u}\to\mathbf{v}\to\mathbf{w}$, which is why the recipe may take $\mathbf{v} = \mathbf{w}\times\mathbf{u}$.

6a. $\mathbf{t}\times\mathbf{w} = (0.8\cdot 0.6 - 0.6\cdot 0.8,\ 0.6\cdot 0 - 1\cdot 0.6,\ 1\cdot 0.8 - 0.8\cdot 0) = (0, -0.6, 0.8)$, already one unit long: $\mathbf{u} = (0, -0.6, 0.8)$, straight down the slope toward the front edge. $\mathbf{v} = \mathbf{w}\times\mathbf{u} = (1, 0, 0)$, along the front edge. $\mathbf{u}\times\mathbf{v} = (0, 0.8, 0.6) = \mathbf{w}$.

6b. $(0, 1, 0)\times\mathbf{w} = (0.6, 0, 0)$, so $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (0, 0.6, -0.8)$, straight up the slope. The pair has turned a quarter turn about $\mathbf{w}$: the new $\mathbf{u}$ is the old $\mathbf{v}$, the new $\mathbf{v}$ is minus the old $\mathbf{u}$.

6c. $(1, 0.8, 0.6) - (1, 0, 0) = \mathbf{w}$, and $\mathbf{w}\times\mathbf{w} = \mathbf{0}$, so both helpers give the same $\mathbf{t}\times\mathbf{w}$. Adding any multiple of $\mathbf{w}$ to a helper changes nothing; only its part across $\mathbf{w}$ matters.

6d. $(0, -4, -3) = -5\mathbf{w}$ and $(0, 1.6, 1.2) = 2\mathbf{w}$ lie along $\mathbf{w}$: $\mathbf{t}\times\mathbf{w} = \mathbf{0}$ and the division fails. $(0, 0, 1)$ works: $\mathbf{u} = (-1, 0, 0)$, $\mathbf{v} = (0, -0.6, 0.8)$, a quarter turn the other way. For $(0, 0.8, 0.61)$, $\mathbf{t}\times\mathbf{w} = (-0.008, 0, 0)$: dividing by 0.008 blows up any rounding error in the numbers 125 times (the book's "low precision").

7.

	view direction	$\mathbf{w}$	$\mathbf{u}$	$\mathbf{v}$
1	$(-3, 0, -4)$	(0.6, 0, 0.8)	(0.8, 0, -0.6)	(0, 1, 0)
2	$(-12, -15, -16)$, length 25	(0.48, 0.6, 0.64)	(0.8, 0, -0.6)	(-0.36, 0.8, -0.48)
3	$(0, -2, 0)$	(0, 1, 0)	fails: $\mathbf{b}\times\mathbf{w} = \mathbf{0}$	—
4	$(0, -2, 0)$	(0, 1, 0)	(1, 0, 0)	(0, 0, -1)

Row 2: $\mathbf{b}\times\mathbf{w} = (0.64,\ 0,\ -0.48)$, length 0.8.

7a. The worked row, row 1 and row 4. In each, the view direction is already perpendicular to up (level cameras; in row 4 up was chosen that way), so there is nothing to straighten.

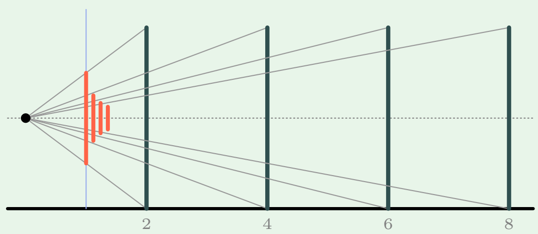
7b. $\mathbf{u}$ is perpendicular to $\mathbf{b} = (0, 1, 0)$, so its y is 0: $\mathbf{u}$ is level. The camera never tips sideways, and the bottom edge of the picture stays parallel to the floor. Rows 1 and 2 are turned the same way and differ only in tilt; $\mathbf{b}\times\mathbf{w} = (w_z,\ 0,\ -w_x)$ uses only the level part of $\mathbf{w}$.

7c. $\text{up}\cdot\mathbf{w} = 0.6$, so $\text{up} - 0.6\,\mathbf{w} = (-0.288,\ 0.64,\ -0.384)$, of length 0.8; divided by 0.8 it is $(-0.36,\ 0.8,\ -0.48) = \mathbf{v}$. Since $\mathbf{v}\cdot\text{up} = 0.8$, $\mathbf{v}$ leans 37° away from up, and the drone looks 37° below the horizon ($\sin = 15/25$): the same angle.

7d. Row 3 fails at $\mathbf{u}$: up lies along the view, so $\mathbf{b}\times\mathbf{w} = \mathbf{0}$ and there is nothing to divide by. Up pointing along the view says nothing about how the picture is turned. Row 4: $\mathbf{v} = (0, 0, -1)$, so the top of the picture shows the floor farther along $-z$, away from you.

7e. $\mathbf{f}\times\text{up} = (-\mathbf{w})\times\mathbf{b} = -(\mathbf{w}\times\mathbf{b}) = \mathbf{b}\times\mathbf{w}$, so right $= \mathbf{u}$. Then right $\times\mathbf{f} = \mathbf{u}\times(-\mathbf{w}) = -(\mathbf{u}\times\mathbf{w}) = \mathbf{w}\times\mathbf{u} = \mathbf{v}$. It is the same recipe, written with $-\mathbf{w}$; no division is needed for up′ because right and $\mathbf{f}$ are perpendicular unit arrows.

8.



images in red, drawn side by side; all lie on the plane $x = 1$

8a,b. Images 1.5, 0.75, 0.5 and 0.375 tall: $3\times 1/2$, $3\times 1/4$, $3\times 1/6$, $3\times 1/8$.

8c. No. The feet are drawn 0.75, 0.375, 0.25 and 0.1875 below eye level: gaps 0.375, 0.125, 0.0625. A post far away has its foot at eye level, on the horizon.

8d. At the middle of every image: the eye is at half the post's height, so the line to the post's middle is level.

8e. Every image is 3 tall and in the same place, so the nearest post hides the other three. Nothing shows the distance.

8f. Images 2.25, 1.125, 0.75, 0.5625. The post at 2 reaches 1.125 above and below eye level and sticks out of the window; the other three fit. A longer d keeps less of the world, as a zoom does.

9a. $(5, 3)\to(1.25,\ 0.75)$; $(2, 1)\to(-0.25,\ -0.25)$; $(4, 0)\to(0.75,\ -0.75)$; $(0.25,\ 0.75)\to(3, 3)$; $(-0.75,\ -0.25)\to(1, 1)$. Backwards, $i = (u - l)/0.5 - 0.5$ and $j = (v - b)/0.5 - 0.5$.

9b. Count whole cells from the corner: $i = \lfloor(1.1+1.5)/0.5\rfloor = \lfloor 5.2\rfloor = 5$ and $j = \lfloor(-0.4+1)/0.5\rfloor = \lfloor 1.2\rfloor = 1$: pixel $(5,1)$.

9c. None. $(0,0)$ is the corner shared by pixels $(2,1)$, $(3,1)$, $(2,2)$ and $(3,2)$. With an even number of columns and rows, no ray goes straight ahead.

9d. Through the window's corner $(l,b) = (-1.5,\ -1)$. Every ray aims half a cell too far left and too low, so every pixel shows what belongs half a pixel down and to its left: the whole picture slides half a pixel up and to the right.

9e. Row $n_y - 1 - j = 4 - 1 - 1 = 2$ of the file.

9f. Cells 0.5 wide and 2/3 tall. A screen draws every pixel square, so heights shrink by $0.5 \div \frac{2}{3} = 0.75$ and a round ball comes out wider than tall. Keep $(r-l)/n_x = (t-b)/n_y$: the window must have the picture's shape.

10.

pixel	(u,v)	$-d\,\mathbf{w}$	$u\,\mathbf{u}$	$v\,\mathbf{v}$	direction
$(0,0)$	$(-1.25,-0.75)$	$(0.8,-0.6,0)$	$(0,0,-1.25)$	$(-0.45,-0.6,0)$	$(0.35,\ -1.2,\ -1.25)$
$(5,1)$	$(1.25,-0.25)$	$(0.8,-0.6,0)$	$(0,0,1.25)$	$(-0.15,-0.2,0)$	$(0.65,\ -0.8,\ 1.25)$
$(2,3)$	$(-0.25,0.75)$	$(0.8,-0.6,0)$	$(0,0,-0.25)$	$(0.45,0.6,0)$	$(1.25,\ 0,\ -0.25)$

10b. $\mathbf{s}-\mathbf{e} = (0.35,-1.2,-1.25)$: dot $\mathbf{u}$ gives -1.25, dot $\mathbf{v}$ gives $0.21-0.96=-0.75$, dot $\mathbf{w}$ gives $-0.28-0.72=-1$. That is $(u,v,-d)$.

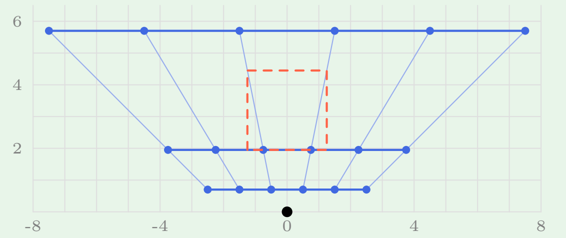
10c. The whole top row, $j=3$: there $v=0.75$ and the direction's y is $-0.6+0.8\times 0.75 = 0$. Those rays run level at the eye's height, toward the horizon.

10d. $d\,\mathbf{w}+u\,\mathbf{u}+v\,\mathbf{v} = (-0.65,\ 0.8,\ 0.75)$: backward (the camera looks along $+x$) and up, out of the back of the camera.

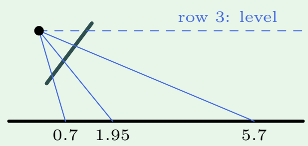
10e. Origin $\mathbf{e}+0.75\,\mathbf{u}+0.25\,\mathbf{v} = (0.15,\ 2.6,\ 0.75)$; direction $-\mathbf{w} = (0.8,\ -0.6,\ 0)$.

11.

j	v	x	y	t at the floor	lands at x
0	-0.75	0.35	-1.2	$2.4/1.2=2$	0.7
1	-0.25	0.65	-0.8	$2.4/0.8=3$	1.95
2	0.25	0.95	-0.4	$2.4/0.4=6$	5.7
3	0.75	1.25	0	never	—



blue: perspective rows 0, 1, 2 dashed red: orthographic



the four row rays, seen from the side

11b. Row 0 lands at $z = \pm0.5, \pm1.5, \pm2.5$; row 1 at $\pm0.75, \pm2.25, \pm3.75$; row 2 at $\pm1.5, \pm4.5, \pm7.5$. A trapezoid, narrow near the camera and wide far away. As a row's rays flatten, t jumps (2, 3, 6), and both the landing distance and the spread $z = t\,u$ grow with it.

11c. The horizon. Row 3's centre rays are level: the upper half of each of its pixels sees the sky, the lower half the far floor.

11d. Row 0: 1 apart, one ray per tile. Row 2: 3 apart, so each ray hits one tile and skips about two. A fine tiled floor far away comes out as a jumble instead of the true pattern (this is aliasing; antialiasing is a later lecture's topic).

11e. Pixel $(0,0)$'s ray starts at $(-0.45,\ 1.8,\ -1.25)$ and meets the floor at $t=3$, at $(1.95,\ 0,\ -1.25)$. In the same way pixel $(5,0)$ lands at $(1.95,\ 0,\ 1.25)$, and pixels $(0,3)$ and $(5,3)$ land at $t=5$, at $(4.45,\ 0,\ -1.25)$ and $(4.45,\ 0,\ 1.25)$. A square 2.5 by 2.5: every row equally wide and the rows equally spaced. Every ray points down along $-\mathbf{w}$, so every row lands: no horizon.

T1. At 20 cm the hole shows less of the room: that is the longer d, and it zooms in.

T2. Similar triangles: $3\times 300/10 = 90$ cm at 10 cm, and 45 cm at 20 cm.

T3. Expect rough agreement. The main error is a moving head: the eye is $\mathbf{e}$, and every line must pass through it.

T4. The thumb, $\mathbf{u}$. The index finger ($\mathbf{v}$) tips forward and the middle finger ($\mathbf{w}$) tips up, as in Exercise 7(b) and 7(c).

12a. $\mathbf{P}-\mathbf{e} = (3.8,\ -1.6,\ 3)$: $u_P = 3$, $v_P = 0.6\cdot3.8+0.8(-1.6) = 2.28-1.28 = 1$, $w_P = -0.8\cdot3.8+0.6(-1.6) = -3.04-0.96 = -4$. In front.

12b. $d/(-w_P) = 1/4$, so $(u,v) = (0.75,\ 0.25)$: pixel $i = \lfloor 2.25/0.5\rfloor = 4$, $j = \lfloor 1.25/0.5\rfloor = 2$, exactly at the centre of $(4,2)$. Its ray: $\mathbf{e}+4\,(0.95,\ -0.4,\ 0.75) = (3.8,\ 0.8,\ 3) = \mathbf{P}$ at $t=4$.

12c. The window is a flat sheet facing the eye, so the similar triangles compare depths measured along $-\mathbf{w}$: the window at depth d, the lantern at depth $-w_P = 4$. The lantern's distance from the eye, $\sqrt{26}\approx 5.1$, also counts how far it is off to the side. Dividing by it would pull every point that is off the centre too far in, the farther off the more, and straight lines would come out bent, as through a fisheye lens.

12d. $\mathbf{P} - \mathbf{e} = (4.8,\ 1.4,\ -1.5)$ gives $(-1.5,\ 4,\ -3)$: in front, at $(u, v) = (-0.5,\ 4/3)$. But $v > t = 1$: above the window's top edge, so it is not in the picture.

12e. $\mathbf{P} - \mathbf{e} = (-1.3,\ 1.6,\ 1.5)$ gives $(1.5,\ 0.5,\ 2)$: $w_P > 0$, behind the camera. Scaling by $d/(-w_P) = -1/2$ anyway gives $(-0.75,\ -0.25)$, pixel $(1, 1)$: the bird would appear, turned upside down and mirrored (both numbers flip sign), although the camera cannot see it.

12f. $(0, -1, 0)$ has camera coordinates $(0,\ -0.8,\ -0.6)$, so the spot is $(0,\ -0.8)/0.6 = (0,\ -4/3)$: below the window's bottom edge ($b = -1$). In a camera tilted down, vertical edges lean toward a point below the picture, so tall things look narrower at the foot. For a level camera, $(0, -1, 0)$ has $w = 0$: there is no such spot, and vertical edges stay parallel in the picture.