Finger Exercises 12

Exact Hits and Soft Shadows

≈ 55 minutes, plus a bonus
CS 453 — Computer Graphics

You need a pencil, a ruler and this sheet. Most exercises ask you to *draw first and compute second.* Every picture is to scale, one grid square per unit, so a careful drawing is accurate to about one decimal, and it tells you what answer to expect before you compute it. Conventions are the lecture's: x right, y up, bold letters are points and arrows, bars around a matrix mean its determinant. Nothing here is an assignment task; the sheet is Lecture 12 only.

Short on time? Do Exercises 2, 3, 6 and 7 (about 25 minutes): the Cramer drill, the what-if table, the shadow construction and the lamp seen from the floor.

Part 1 · A ray meets a wall

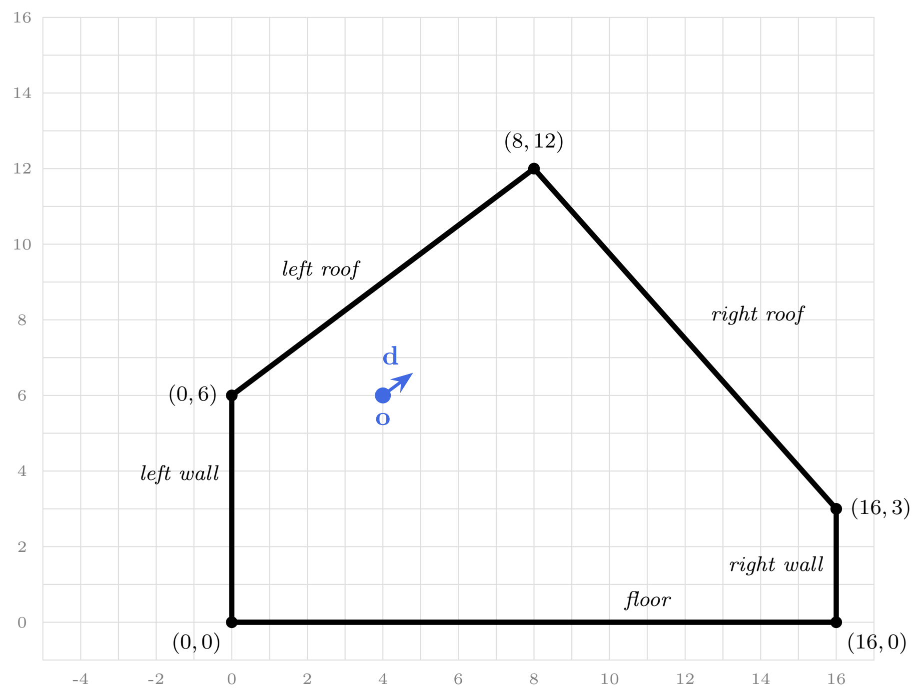
Exercises 1–5

A room shaped like a house has five walls. A ray leaves the eye $\mathbf{o} = (4,\ 6)$ along the unit arrow $\mathbf{d} = (0.8,\ 0.6)$.

1 Extend everything

about 5 min

See t and u on the picture before any algebra: every wall's line crosses the ray's line somewhere, and only the picture says which crossings are real.



(a) With a ruler, extend the ray forwards to the edge of the grid, and backwards, dashed, to the edge of the grid.

(b) Extend each wall, lightly, into a full line. Circle every point where the ray's line crosses a wall's line. Which wall's line is never crossed?

(c) For each circle: is it *ahead* of $\mathbf{o}$ or *behind* it? Is it *on* the wall, or on the extension *past* one of the wall's ends?

(d) Exactly one circle is a real hit. Read its coordinates off the grid. From them, estimate t (how many steps of $\mathbf{d}$, each 0.8 across, from $\mathbf{o}$) and u (the fraction of the way along that wall, measured from its end at $(16, 3)$).

2 One ray, five walls

about 8 min

Cramer's rule until it is automatic, and the three checks that turn two numbers into hit or miss.

For a wall from $\mathbf{a}$ to $\mathbf{b}$: $\mathbf{A} = [\,\mathbf{d}\ \ \mathbf{a}-\mathbf{b}\,]$ (two columns), right side $\mathbf{a}-\mathbf{o}$, and $\left|\begin{smallmatrix} p & q \\ r & s \end{smallmatrix}\right| = ps - qr$. $t = |\mathbf{A}_t|/|\mathbf{A}|$ with $\mathbf{a}-\mathbf{o}$ in the first column; $u = |\mathbf{A}_u|/|\mathbf{A}|$ with $\mathbf{a}-\mathbf{o}$ in the second. A hit needs $|\mathbf{A}| \neq 0$, $t \geq 0$ and $0 \leq u \leq 1$. The first row is done for you.

wall, $\mathbf{a} \to \mathbf{b}$	$\mathbf{a}-\mathbf{b}$	$\mathbf{a}-\mathbf{o}$	$\lvert\mathbf{A}\rvert$	$\lvert\mathbf{A}_t\rvert$	$\lvert\mathbf{A}_u\rvert$	t	u	verdict
floor $(0,0) \to (16,0)$	$(-16, 0)$	$(-4, -6)$	9.6	-96	-2.4	-10	-0.25	miss: behind, and past $\mathbf{a}$
right wall $(16,0) \to (16,3)$	______	______	______	______	______	______	______	______
right roof $(16,3) \to (8,12)$	______	______	______	______	______	______	______	______
left roof $(8,12) \to (0,6)$	______	______	______	______	______	______	______	______
left wall $(0,6) \to (0,0)$	______	______	______	______	______	______	______	______

(a) Compare each row with your circles from Exercise 1. Did the numbers overturn any of your predictions?

(b) Which wall does the eye see, and at which point?

(c) The right wall and the right roof both have a negative $|\mathbf{A}|$; one is a hit and one is not. What does the sign of $|\mathbf{A}|$ tell you about hit or miss?

3 What if . . . ?

about 5 min

Predict how t, u and $|\mathbf{A}|$ respond to a change from the shape of Cramer's rule, without recomputing.

Start from the right roof's row: $|\mathbf{A}| = -12$, $t = 7$, $u = 0.8$. Make one change at a time and predict the new values. For each, ask which ingredient changed: t's column $\mathbf{d}$, u's column $\mathbf{a}-\mathbf{b}$, or the right side $\mathbf{a}-\mathbf{o}$.

	change	$\lvert\mathbf{A}\rvert$	t	u	hit?
	none: the right roof as in Exercise 2	-12	7	0.8	yes
1	the eye moves 2 units forward along $\mathbf{d}$, to $(5.6,\ 7.2)$	______	______	______	______
2	$\mathbf{d}$ is doubled, to $(1.6,\ 1.2)$	______	______	______	______
3	$\mathbf{d}$ is reversed, to $(-0.8,\ -0.6)$	______	______	______	______
4	the roof's ends are named the other way round: $\mathbf{a} = (8, 12)$, $\mathbf{b} = (16, 3)$	______	______	______	______
5	the whole room and the eye move 3 right and 1 down	______	______	______	______
6	the roof is made twice as long past its upper end: $\mathbf{b} = (0,\ 21)$	______	______	______	______

(a) Check rows 3 and 6 with Cramer's rule.

(b) Row 2 gives $t = 3.5$. How far from the eye is the hit?

(c) Row 3's ray looks backwards. Using your Exercise 2 table and no new computation, say which wall it sees and at what t.

4 Build to order

about 5 min

Run the idea backwards: choose the ray, or the wall, that produces a required answer.

(a) Where on the floor must a ray with $\mathbf{d} = (0.8,\ 0.6)$ start so that it hits the right roof exactly at its middle? How far does it travel? Check with Cramer's rule.

(b) From the eye $\mathbf{o} = (4, 6)$, find the unit direction whose ray lands on the floor at $(8.5,\ 0)$. What are t and u there?

(c) Add a vertical shelf, 4 long, that the eye's ray (with $\mathbf{d} = (0.8, 0.6)$) hits at $t = 3$, a quarter of the way down from its top. Give its top end $\mathbf{a}$ and bottom end $\mathbf{b}$, then check with Cramer's rule.

(d) Give a unit direction from the eye for which Cramer's rule cannot be used on the floor, and one for which it cannot be used on the left wall.

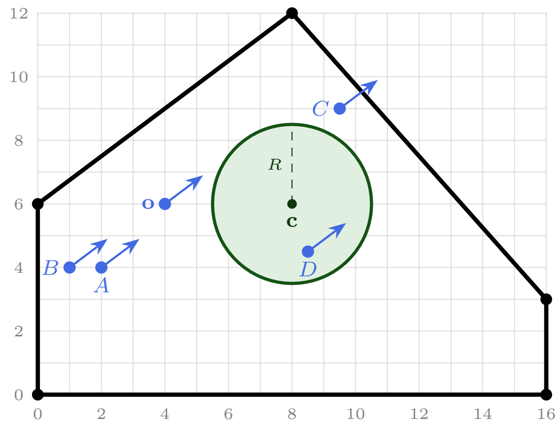
5 A ball in the room

about 7 min

The circle's quadratic, and deciding hit or miss from the signs of β and γ before taking any square root.

A ball hangs in the room: centre $\mathbf{c} = (8,\ 6)$, radius $R = 2.5$ (the slide calls the radius r). Every ray below points along $\mathbf{d} = (0.8,\ 0.6)$, and

$$\beta = 2\,\mathbf{d}\cdot(\mathbf{o}-\mathbf{c}), \qquad \gamma = \|\mathbf{o}-\mathbf{c}\|^2 - R^2, \qquad t = \tfrac{1}{2}\big(-\beta \pm \sqrt{\beta^2 - 4\gamma}\big).$$



$\mathbf{o} = (4, 6)$, $A = (2, 4)$, $B = (1, 4)$, $C = (9.5, 9)$, $D = (8.5, 4.5)$; the arrows show $\mathbf{d}$'s direction only.

(a) The eye's ray, from $\mathbf{o} = (4, 6)$: find β, γ, $\beta^2 - 4\gamma$, both roots, and the hit point.

(b) Exercise 2 found the right roof at $t = 7$. What does the eye really see?

(c) **Decide before you take a root.** $\gamma < 0$ exactly when the start is inside the ball; $\beta < 0$ exactly when the ball's centre is ahead of the start. For each start, decide from the signs whether the square root is needed at all, then finish.

start	β	γ	inside?	centre ahead?	$\beta^2 - 4\gamma$	verdict, and t
A $(2,\ 4)$	____	____	____	____	____	________
B $(1,\ 4)$	____	____	____	____	____	________
C $(9.5,\ 9)$	____	____	____	____	____	________
D $(8.5,\ 4.5)$	____	____	____	____	____	________

(d) The two roots are where the ray enters and leaves the ball. Their average is the ray's point closest to $\mathbf{c}$; half their difference is half the chord. From the eye's roots, how close does its ray pass to $\mathbf{c}$? Why does it only clip the ball?

Part 2 · Soft shadows

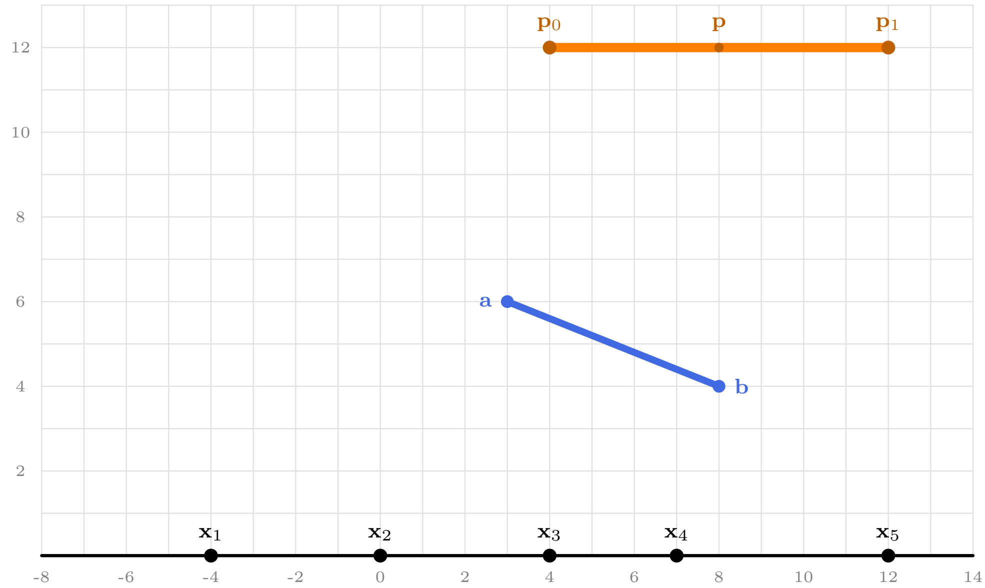
Exercises 6–10

A plank runs from $\mathbf{a} = (3,\ 6)$ to $\mathbf{b} = (8,\ 4)$ above the floor $y = 0$. A tube light runs from $\mathbf{p}_0 = (4,\ 12)$ to $\mathbf{p}_1 = (12,\ 12)$: it is 8 long, with its centre at $\mathbf{p} = (8,\ 12)$. Five floor points: $\mathbf{x}_1 = (-4, 0)$, $\mathbf{x}_2 = (0, 0)$, $\mathbf{x}_3 = (4, 0)$, $\mathbf{x}_4 = (7, 0)$, $\mathbf{x}_5 = (12, 0)$.

6 Draw the shadow

about 5 min

Construct hard and soft shadows with a ruler. Exercises 7, 8 and 10 read their answers off this drawing.



(a) Shrink the tube to its centre $\mathbf{p}$: a point light. Draw the lines from $\mathbf{p}$ through $\mathbf{a}$ and through $\mathbf{b}$ down to the floor. Check where they land by counting steps: $\mathbf{a} - \mathbf{p} = (-5, -6)$ falls 6 per step, $\mathbf{b} - \mathbf{p} = (0, -8)$ falls 8. Shade the hard shadow.

(b) Now the whole tube. Draw the four lines from $\mathbf{p}_0$ and from $\mathbf{p}_1$ through $\mathbf{a}$ and through $\mathbf{b}$. Where does each land? Shade dark where both ends of the tube are hidden (the umbra), and light where only one is (the two penumbras).

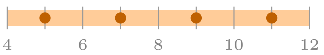
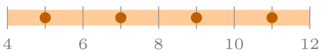
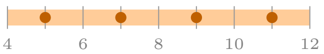
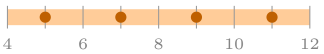
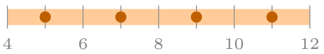
(c) Where do the point light's two edges fall inside your zones? Is the umbra wider or narrower than the hard shadow?

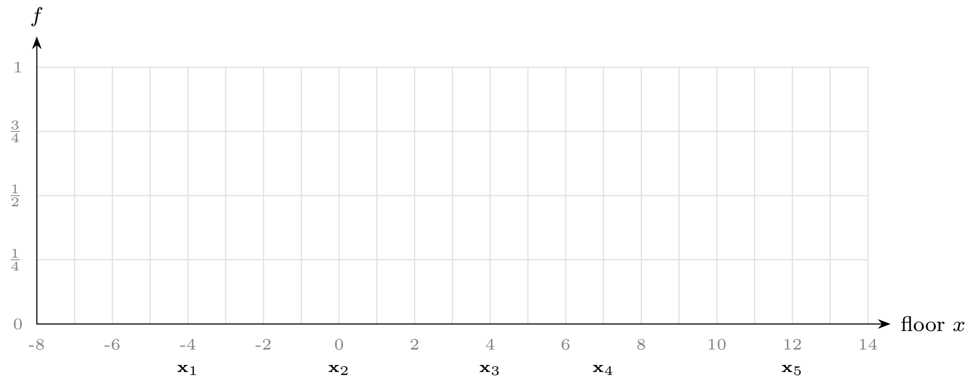
7 What the floor sees

about 8 min

Visibility as a fraction: the tube as seen from each floor point, first exactly, then with four shadow rays.

Stand at a floor point and look up: the plank hides part of the tube. The hidden part ends on the line from the floor point through one end of the plank: through **a** for points left of the umbra, through **b** for points right of it. Draw that line on your Exercise 6 picture, carry it up to $y = 12$, and hatch the hidden part of the tube on the bar. The four dots are the samples: the centres of four equal pieces of the tube.

floor point	line meets $y = 12$ at	the tube from there	exact f	samples in view	$k/4$
$\mathbf{x}_1 = (-4, 0)$	______		______	______	____
$\mathbf{x}_2 = (0, 0)$	______		______	______	____
$\mathbf{x}_3 = (4, 0)$	______		______	______	____
$\mathbf{x}_4 = (7, 0)$	______		______	______	____
$\mathbf{x}_5 = (12, 0)$	______		______	______	____



(a) Plot the exact f for $\mathbf{x}_1, \ldots, \mathbf{x}_5$ and at the zone edges -6, 2, 6 and 10, and join the points with straight lines. Why straight?

(b) With four samples, f can only be $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. It jumps where the line through **a** (or **b**) passes exactly over a sample. Find the eight jump positions and draw the staircase on the same axes.

(c) Where are the staircase and the exact f furthest apart, and by how much?

(d) Lecture 1: a colour channel has 256 levels. How many samples keep that gap below one level?

8 Spread the rays

about 4 min

Why spreading rays over a pixel costs nothing extra and still measures the lamp.

A pixel sends four viewing rays that land on the floor at $x = -3.5$, -2.5, -1.5 and -0.5 (much wider apart than in a real pixel, so they can be drawn). Instead of four shadow rays each, every viewing ray sends *one* shadow ray, to its own sample.

viewing ray lands at	its sample	line through $\mathbf{a}$ meets $y = 12$ at	sample in view?
-3.5	5	________	________
-2.5	7	________	________
-1.5	9	________	________
-0.5	11	________	________

(a) What f does the pixel get from these four shadow rays?

(b) Now *add* instead: each of the four floor points sends all four shadow rays. How many rays is that, and what f?

(c) The pixel's true value is the average of the exact f at the four points. Compare all three answers.

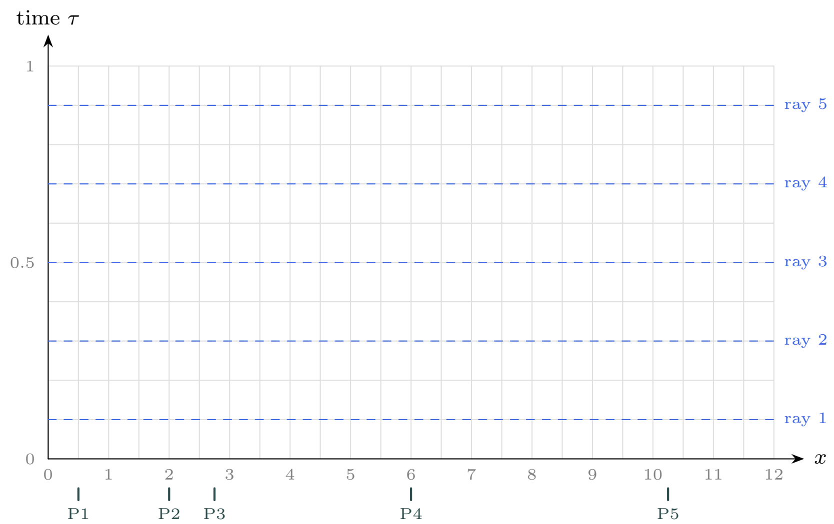
(d) Suppose all four viewing rays use the same sample, 5. What f do you get, and what kind of light have you built?

9 Motion blur in space and time

about 5 min

The same idea spread over time, drawn as a space-time diagram.

A box 1.5 wide slides right while the shutter is open. At time $\tau = 0$ it covers $0 \le x \le 1.5$; at $\tau = 1$ it covers $10 \le x \le 11.5$. Position runs across the diagram and time runs up. Each pixel fires five rays, at the middles of five equal pieces of the shutter time (the dashed lines).



(a) Draw the box at $\tau = 0$ and at $\tau = 1$, and shade the band it sweeps between them.

(b) Pixel columns P1 to P5 sit at $x = 0.5$, 2, 2.75, 6 and 10.25. For each, read how long the box covers it and which rays see it.

	P1	P2	P3	P4	P5
share of the time covered	______	______	______	______	______
rays that see the box	______	______	______	______	______
pixel value $k/5$	______	______	______	______	______

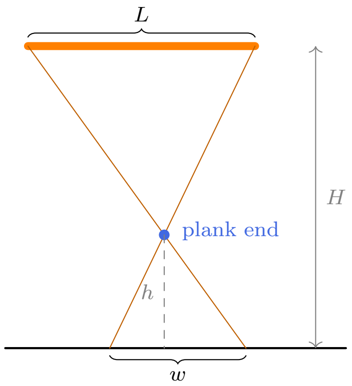
(c) Along the row of pixels, the five rays paint five faint copies of the box. Where are they, and where are the gaps between them?

(d) The box moves $10/N$ between two of N rays. How many rays per pixel close the gaps?

10 Shadow detective

about 3 min

Read a scene backwards from the width of its soft edges.



(a) The lines from the tube's ends through a plank end cross at that end. The triangle above the crossing (base L, height $H - h$) and the one below it (base w, height h) are similar. Show that $w = L\,h/(H - h)$, and check it against both penumbras of Exercise 6.

(b) Under the same tube, a shelf edge casts a penumbra 2 wide. How high is the edge?

(c) Another tube, also 12 up: an edge 3 above the floor casts a penumbra 3 wide. How long is that tube?

(d) The edge rests on the floor. What does the formula give?

Try it tonight — optional, about 10 minutes, no marks

You need a tube light, a book, a ruler and your phone.

1. Hold the book flat about 20 cm above a table, under the tube, with two of its edges along the tube and two across it. Two of the shadow's edges are soft and two are nearly sharp. Which two are soft, and why? (In which direction is the light large?)
2. Measure the width of a soft edge with the book about 5 cm and about 20 cm above the table. Compare with $w = L\,h/(H - h)$, where L is the tube's length (a standard 4-foot tube is about 120 cm) and H its height above the table.
3. Lower the book onto the table. What happens to the soft edges?
4. Switch the tube off and light the book with your phone's torch. Why are all four edges sharp now?

11 Bonus: what Cramer's rule is measuring ★

about 10 min

For the brave. Back to the room of Part 1: $\mathbf{o} = (4, 6)$, $\mathbf{d} = (0.8,\ 0.6)$.

Because $\mathbf{d}$ is a unit arrow, the single number $\begin{vmatrix} d_x & p_x - o_x \\ d_y & p_y - o_y \end{vmatrix}$ is the *signed distance* of a point $\mathbf{p}$ from the ray's line: positive when $\mathbf{p}$ lies to the left of $\mathbf{d}$, negative when it lies to the right.

(a) Find this distance for both ends of the right roof: s_a for $\mathbf{a} = (16, 3)$ and s_b for $\mathbf{b} = (8, 12)$. Find s_a, and $s_a - s_b$, in your Exercise 2 row.

(b) So $u = s_a/(s_a - s_b)$. Explain it with lecture 5's lerp: walking from $\mathbf{a}$ to $\mathbf{b}$, the distance changes steadily from s_a to s_b. Where does it pass zero?

(c) Fill the table for all five walls. Which walls have their two ends on opposite sides of the ray's line? Compare with the walls whose u lies in $[0, 1]$. What do the left roof's two numbers say?

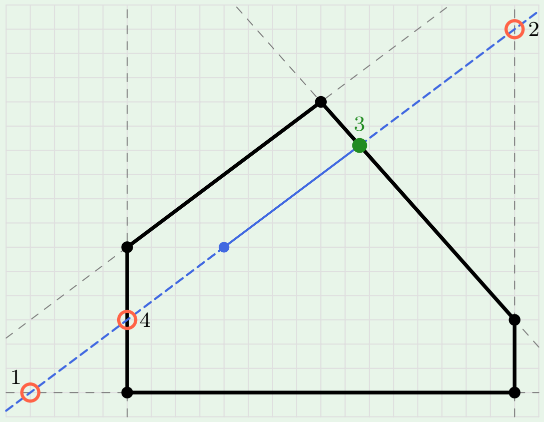
	floor	right wall	right roof	left roof	left wall
s_a	______	______	______	______	______
s_b	______	______	______	______	______
opposite sides?	______	______	______	______	______

(d) Now t. The left wall is the line $x = 0$. How far is $\mathbf{o}$ from it, and how fast does the ray's x change per unit of t? Divide to get t. Then show that the left wall's $|\mathbf{A}_t|$ and $|\mathbf{A}|$ are these two numbers, each multiplied by the wall's length 6.

Answer Key

Check your work *after* attempting everything. If an answer surprises you, that item is worth re-reading in the slides.

1.



1b. Four crossings. The left roof's line is never crossed: it is parallel to $\mathbf{d}$ (the ray runs 2.4 below it).

1c. **1** floor line at $(-4, 0)$: behind, past the end $(0, 0)$. **2** right wall's line at $(16, 15)$: ahead, past the top end $(16, 3)$. **3** right roof at $(9.6,\ 10.2)$: ahead, on the roof — the hit. **4** left wall at $(0, 3)$: on the wall, but behind.

1d. From $(9.6,\ 10.2)$: $t = (9.6 - 4)/0.8 = 7$, and $u = (16 - 9.6)/8 = 0.8$.

2.

wall	$\mathbf{a} - \mathbf{b}$	$\mathbf{a} - \mathbf{o}$	$\lvert\mathbf{A}\rvert$	$\lvert\mathbf{A}_t\rvert$	$\lvert\mathbf{A}_u\rvert$	t	u	verdict
floor	$(-16, 0)$	$(-4, -6)$	9.6	-96	-2.4	-10	-0.25	miss: behind, past **a**
right wall	$(0, -3)$	$(12, -6)$	-2.4	-36	-12	15	5	miss: past **b**
right roof	$(8, -9)$	$(12, -3)$	-12	-84	-9.6	7	0.8	**hit**
left roof	$(8, 6)$	$(4, 6)$	0	—	—	—	—	miss: parallel
left wall	$(0, 6)$	$(-4, 0)$	4.8	-24	2.4	-5	0.5	miss: behind

2a. They should agree: every t and u matches a circle from Exercise 1.

2b. The right roof, at $(9.6,\ 10.2)$, 7 units away.

2c. Nothing. Both signs are negative, yet one is a hit and one a miss. The sign changes if we swap which end is called **a** (Exercise 3, row 4). Only $|\mathbf{A}| = 0$ matters.

3.

	$\lvert\mathbf{A}\rvert$	t	u	hit?	why
1	-12	5	0.8	yes	only the right side changed; the start is 2 units nearer
2	-24	3.5	0.8	yes	t's column doubled, so $\lvert\mathbf{A}\rvert$ doubles and t halves
3	12	-7	0.8	no	t's column negated: $\lvert\mathbf{A}\rvert$ and t flip sign; now behind
4	12	7	0.2	yes	u's column is negated *and* $\mathbf{a} - \mathbf{o}$ changes: the same point, with u now measured from the other end, $1 - 0.8$
5	-12	7	0.8	yes	$\mathbf{d}$, $\mathbf{a} - \mathbf{b}$ and $\mathbf{a} - \mathbf{o}$ are all differences, so nothing changes
6	-24	7	0.4	yes	u's column doubled: the same point is 0.4 of a wall twice as long

3a. Row 3: $|\mathbf{A}| = (-0.8)(-9) - 8(-0.6) = 12$, $|\mathbf{A}_t| = -84$, $|\mathbf{A}_u| = (-0.8)(-3) - 12(-0.6) = 9.6$, so $t = -7$, $u = 0.8$. Row 6: $\mathbf{a} - \mathbf{b} = (16, -18)$, $|\mathbf{A}| = -14.4 - 9.6 = -24$, $|\mathbf{A}_t| = 12(-18) - 16(-3) = -168$, $|\mathbf{A}_u| = -9.6$, so $t = 7$, $u = 0.4$.

3b. 7 units: 3.5 steps of an arrow 2 long.

3c. Reversing $\mathbf{d}$ flips every t and keeps every u. The left wall's $(-5,\ 0.5)$ becomes $(5,\ 0.5)$: the reversed ray sees the left wall at $(0, 3)$, 5 units away. The floor turns into $t = 10$, but its $u = -0.25$ still misses.

4a. The roof's middle is $(16, 3) + 0.5\,(-8, 9) = (12,\ 7.5)$. From the floor it takes $7.5/0.6 = 12.5$ steps, so the ray starts at $12 - 12.5 \cdot 0.8 = 2$: the start is $(2,\ 0)$ and $t = 12.5$. Cramer: $\mathbf{a} - \mathbf{o} = (14, 3)$, $|\mathbf{A}_t| = 14(-9) - 8 \cdot 3 = -150$, $|\mathbf{A}_u| = 0.8 \cdot 3 - 14 \cdot 0.6 = -6$, $|\mathbf{A}| = -12$, so $t = 12.5$ and $u = 0.5$.

4b. $(8.5, 0) - (4, 6) = (4.5,\ -6)$, of length 7.5, so $\mathbf{d} = (0.6,\ -0.8)$ and $t = 7.5$; $u = 8.5/16 = 0.53125$. Cramer: $|\mathbf{A}| = -12.8$, $|\mathbf{A}_t| = -96$, $|\mathbf{A}_u| = -6.8$.

4c. The hit point is $\mathbf{o} + 3\mathbf{d} = (6.4,\ 7.8)$, and a quarter of 4 below the top gives $\mathbf{a} = (6.4,\ 8.8)$, $\mathbf{b} = (6.4,\ 4.8)$. Cramer: $\mathbf{a} - \mathbf{b} = (0, 4)$, $\mathbf{a} - \mathbf{o} = (2.4, 2.8)$, $|\mathbf{A}| = 3.2$, $|\mathbf{A}_t| = 9.6$, $|\mathbf{A}_u| = 2.24 - 1.44 = 0.8$: $t = 3$, $u = 0.25$. The shelf is now nearer than the roof.

4d. The floor: $(1, 0)$ or $(-1, 0)$. The left wall: $(0, 1)$ or $(0, -1)$. Either way the ray is parallel to that wall and $|\mathbf{A}| = 0$.

5a. $\mathbf{o} - \mathbf{c} = (-4, 0)$: $\beta = 2(-3.2) = -6.4$, $\gamma = 16 - 6.25 = 9.75$, $\beta^2 - 4\gamma = 40.96 - 39 = 1.96$, whose root is 1.4. The roots are $t = (6.4 \pm 1.4)/2$, that is 2.5 and 3.9. The hit is at $t = 2.5$, the point $(6,\ 7.5)$, which is 2.5 from $\mathbf{c}$.

5b. The ball: $2.5 < 7$. The nearest hit over every object, walls and ball alike, is what the eye sees.

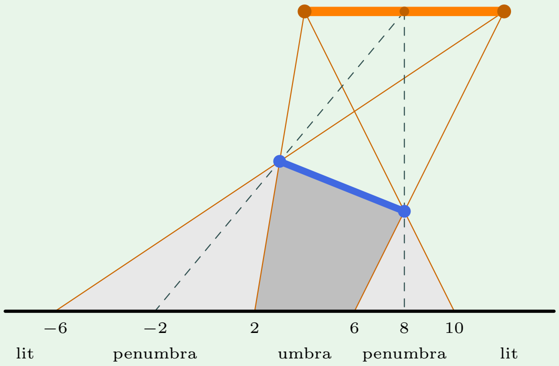
5c.

start	β	γ	inside?	centre ahead?	$\beta^2 - 4\gamma$	verdict, and t
A $(2, 4)$	-12	33.75	no	yes	9	hit at $t = 4.5$, (5.6, 6.7)
B $(1, 4)$	-13.6	46.75	no	yes	-2.04	miss: passes just above the ball
C $(9.5, 9)$	6	5	no	no	not needed	miss: the ball is behind
D $(8.5, 4.5)$	-1	-3.75	yes	—	16	hit, the larger root: $t = 2.5$, (10.5, 6)

The rule: inside always means a hit, at the larger root. Outside with the centre behind means a miss. Outside with the centre ahead, the discriminant decides. (The roots add to $-\beta$ and multiply to γ, which is why the signs are enough.)

5d. The average is $(2.5+3.9)/2 = 3.2$ and half the difference is 0.7, so the closest approach is $\sqrt{2.5^2 - 0.7^2} = \sqrt{5.76} = 2.4$. That is only 0.1 inside the surface, so the chord is short.

6.



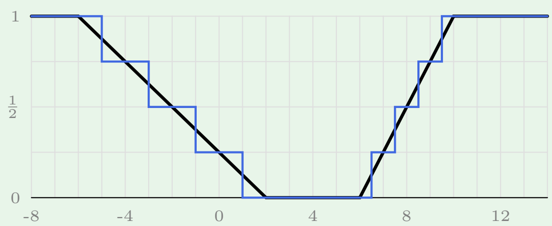
6a. Through **a**: 2 steps, landing at $(-2, 0)$. Through **b**: 1.5 steps, landing at $(8, 0)$. Hard shadow: $-2 \le x \le 8$ (dashed).

6b. $\mathbf{p}_0$: through **a**, step $(-1, -6)$, lands at 2; through **b**, step $(4, -8)$, lands at 10. $\mathbf{p}_1$: through **a**, step $(-9, -6)$, lands at -6; through **b**, step $(-4, -8)$, lands at 6. Umbra $2 \le x \le 6$; penumbras $-6 \le x \le 2$ and $6 \le x \le 10$.

6c. -2 and 8 are the *middles* of the two penumbras. The umbra (4 wide) is much narrower than the hard shadow (10 wide).

7.

point	line meets $y = 12$ at	hidden part	exact f	samples in view	$k/4$
$\mathbf{x}_1$	10 (via **a**)	10 to 12	$6/8 = 3/4$	$5, 7, 9$	$3/4$
$\mathbf{x}_2$	6 (via **a**)	6 to 12	$2/8 = 1/4$	5	$1/4$
$\mathbf{x}_3$	2 (via **a**)	all of it	0	none	0
$\mathbf{x}_4$	10 (via **b**)	4 to 10	$2/8 = 1/4$	11	$1/4$
$\mathbf{x}_5$	0 (via **b**)	none	1	all four	1



black: exact f blue: four samples

7a. Straight because the meeting point, $6-x$ on the left and $24-2x$ on the right, moves steadily with x.

7b. Left: $x = -5, -3, -1, 1$ (where $6-x$ is $11, 9, 7, 5$). Right: $x = 6.5, 7.5, 8.5, 9.5$ (where $24-2x$ is $5, 7, 9, 11$).

7c. Just beside each jump, by $1/8 = 1/(2N)$. At the five floor points the two agree exactly.

7d. $1/(2N) \le 1/256$ needs $N \ge 128$.

8. The meeting points are 9.5, 8.5, 7.5 and 6.5, so samples 5 and 7 are in view and 9 and 11 are not.

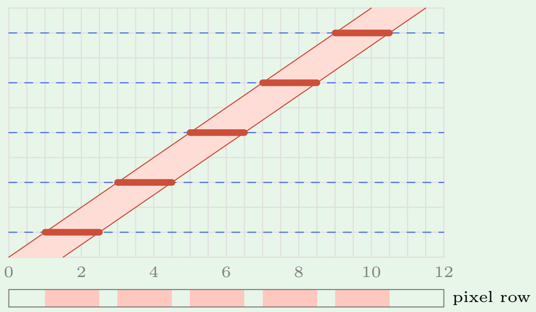
8a. $2/4 = 1/2$.

8b. 16 rays, and $3+2+2+1 = 8$ arrive: $1/2$.

8c. The exact values are $5.5/8$, $4.5/8$, $3.5/8$, $2.5/8$, averaging $1/2$. All three agree, and spreading used 4 shadow rays instead of 16.

8d. Sample 5 is in view from all four points: $f = 1$. That is a point light at $(5, 12)$, and its shadow is hard again. Spreading works only if every viewing ray aims at a different part of the lamp.

9.



9b.

	P1	P2	P3	P4	P5
covered	0.05	0.15	0.15	0.15	0.125
rays	none	ray 1	none	ray 3	ray 5
value	0	$1/5$	0	$1/5$	$1/5$

9c. Copies at $[1, 2.5]$, $[3, 4.5]$, $[5, 6.5]$, $[7, 8.5]$ and $[9, 10.5]$, each $1/5$ strength. The gaps are $(2.5, 3)$, $(4.5, 5)$, $(6.5, 7)$ and $(8.5, 9)$. A pixel in a gap, like P3, shows no box at all, though the box covered it for 0.15 of the time.

9d. The gaps close when $10/N \le 1.5$: $N \ge 7$.

10a. Similar triangles give $w/L = h/(H-h)$, so $w = L\,h/(H-h)$. At **a**: $8 \cdot 6/6 = 8$ (from -6 to 2). At **b**: $8 \cdot 4/8 = 4$ (from 6 to 10).

10b. $2 = 8h/(12-h)$ gives $24 - 2h = 8h$, so $h = 2.4$.

10c. $L = w(H - h)/h = 3 \cdot 9/3 = 9$.

10d. $w = 0$: a sharp contact edge, as under a table leg.

T1. The edges running *across* the tube are soft. Along its length the tube is about 120 cm wide, but across it the tube is only about 3 cm thick, so edges running along the tube stay nearly sharp.

T2. With $H \approx 200$ cm: $h = 5$ gives $w \approx 120 \cdot 5/195 \approx 3$ cm, and $h = 20$ gives $w \approx 13$ cm. Expect rough agreement; other lights in the room blur the picture.

T3. The soft edges shrink to nothing.

T4. The torch is almost a point, so every edge is sharp.

11a. $s_a = 0.8(-3) - 0.6(12) = -9.6$ and $s_b = 0.8(6) - 0.6(4) = 2.4$. In Exercise 2, $|\mathbf{A}_u| = -9.6 = s_a$ and $|\mathbf{A}| = -12 = s_a - s_b$.

11b. Along the roof the distance is $s_a + u\,(s_b - s_a) = -9.6 + 12u$, which is zero at $u = 0.8$: the crossing.

11c. Floor $-2.4, -12$; right wall $-12, -9.6$; right roof $-9.6, 2.4$; left roof $2.4, 2.4$; left wall $2.4, -2.4$. Only the right roof and the left wall have their ends on opposite sides, and they are exactly the walls with u in $[0, 1]$. The sign of t then says ahead (the roof) or behind (the left wall). This test needs no division, so it can reject most walls before any is done. The left roof's ends are *equally* far from the line, so the roof runs parallel to it and $|\mathbf{A}| = s_a - s_b = 0$.

11d. $\mathbf{o}$ is 4 from the line $x = 0$, and the ray's x grows by 0.8 per unit of t, away from the wall. So $t = 4/(-0.8) = -5$. Cramer: $|\mathbf{A}_t| = -24 = -4 \times 6$ and $|\mathbf{A}| = 4.8 = 0.8 \times 6$. The wall's length cancels in the ratio.