Finger Exercises 7

2×2 Matrices: Area, Scale, Shear, Rotation

≈ 60 minutes, plus a 10-minute bonus

CS 453 — Computer Graphics

What this is: Short drills that lock in Lecture 7 — the determinant as a signed area, the rules for multiplying, undoing and transposing matrices, the two ways to read a matrix times a vector, and the three moves every later lecture builds on: scale, shear and rotation.

How to do it: On **paper** first, no computer. Keep squared paper next to you: sketch every arrow and every clock *before* you multiply, and predict the picture in words. Then let the arithmetic confirm or correct you. Conventions are the book's: x right, y up, positive angles counterclockwise, column vectors with the matrix on the left, $|\mathbf{ab}|$ for a determinant. If an item takes longer than 5 minutes, skip it, finish the rest, come back.

Part A • Do You Remember? RECALL ≈ 5 min

A1. $|\mathbf{ab}|$ is the ________________ of the parallelogram built on $\mathbf{a}$ and $\mathbf{b}$. It is positive when the turn from $\mathbf{a}$ to $\mathbf{b}$ (through the smaller angle) is ______________, negative when it is ____________, and zero when ____________________.

A2. In coordinates, $|\mathbf{ab}|$ = ____________________. Name the three picture rules the formula was built from: _______________, _______________, _______________.

A3. If $\mathbf{A}$ is $m \times n$, the product $\mathbf{AB}$ exists only when $\mathbf{B}$ has ____________, and the answer is __________. Entry p_{ij} walks along ____________ of $\mathbf{A}$ and down ____________ of $\mathbf{B}$.

A4. Fill in = or "not always =": $\mathbf{AB}$ __________ $\mathbf{BA}$; $(\mathbf{AB})\mathbf{C}$ __________ $\mathbf{A}(\mathbf{BC})$; $\mathbf{AA}^{-1}$ __________ $\mathbf{A}^{-1}\mathbf{A}$.

A5. $(\mathbf{AB})^{-1}$ = ___________, $(\mathbf{AB})^{\mathrm{T}}$ = ___________, $|\mathbf{AB}|$ = __________, $|\mathbf{A}^{-1}|$ = ________, $|\mathbf{A}^{\mathrm{T}}|$ = ________.

A6. An *orthogonal* matrix has columns that are _____________ and _____________. Therefore $\mathbf{R}^{\mathrm{T}}\mathbf{R}$ = _____, its inverse is ____________, and its determinant is __________.

A7. Write from memory: scale(s_x, s_y) = _________, shear-$x(s)$ = _________, rotate(ϕ) = _____________.

A8. The one sentence to keep from Lecture 7: the columns of a matrix are ______________________________. And one thing *no* 2×2 matrix can ever do to a point is _______________________.

Part B • Signed Area on the Grid COMPUTE ≈ 6 min

B1. Take $\mathbf{a} = (4, 1)$ and $\mathbf{b} = (1, 3)$. Compute $|\mathbf{ab}|$, and say *from your sketch* why the sign is right.

B2. Check your 11 without the formula. Draw the parallelogram with corners $(0, 0)$, $(4, 1)$, $(5, 4)$, $(1, 3)$, put the smallest box around it, and subtract the corner pieces.

B3. Without any arithmetic: $|\mathbf{ba}|$ = __________. Why?

B4. $\mathbf{c} = (2, -1)$ and $\mathbf{d} = (-4, 2)$. Compute $|\mathbf{cd}|$ and explain the answer with a picture.

B5. $\mathbf{u} = (-2, -1)$ and $\mathbf{v} = (2, -3)$ both point *downward*. A classmate says "so $|\mathbf{uv}|$ must be negative". Predict the sign from a sketch, then compute.

B6. For $\mathbf{a} = (3, k)$ and $\mathbf{b} = (6, 4)$: which k makes $|\mathbf{ab}| = 0$? Which k makes it $+9$?

Part C • Area Rules, No Coordinates COMPUTE ≈ 5 min

Use $|\mathbf{ab}| = 11$ from B1, but **do not** use the coordinates of $\mathbf{a}$ and $\mathbf{b}$ until C5. Name the rule you use each time.

C1. $|(3\mathbf{a})\,\mathbf{b}|$ = __________ $|(-\mathbf{a})\,(2\mathbf{b})|$ = __________

C2. $|\mathbf{a}\,(\mathbf{b} - 4\mathbf{a})|$ = __________. Draw a one-line picture of why.

C3. A classmate writes $|(2\mathbf{a})(2\mathbf{b})| = 2|\mathbf{ab}| = 22$. What is the right value, and what is the picture?

C4. Expand $|(\mathbf{a} + \mathbf{b})(\mathbf{a} - \mathbf{b})|$ into four determinants, then simplify to a number.

C5. Now check C4 with coordinates from B1.

Part D • Row Meets Column

COMPUTE ≈ 6 min

D1. Give the size of each product, or write "not defined".
$(2\times3)(3\times1)$: ________ $(3\times2)(3\times2)$: ________ $(1\times2)(2\times1)$: ________
$(2\times1)(1\times2)$: ________ $(2\times2)(2\times3)$: ________

D2. Compute $\begin{bmatrix}1 & 0 & 2\\ -1 & 3 & 1\end{bmatrix}\begin{bmatrix}1\\2\\-1\end{bmatrix}$.

D3. Let $\mathbf{A}=\begin{bmatrix}2 & 1\\ 0 & 3\end{bmatrix}$ and $\mathbf{B}=\begin{bmatrix}1 & -1\\ 2 & 0\end{bmatrix}$. Compute $\mathbf{AB}$, one entry at a time.

D4. Now compute $\mathbf{BA}$. Same matrix?

D5. Compute $|\mathbf{A}|$, $|\mathbf{B}|$, $|\mathbf{AB}|$ and $|\mathbf{BA}|$. The two products were different matrices; why do their determinants agree?

D6. $\mathbf{S}=\begin{bmatrix}2 & 0\\ 0 & 5\end{bmatrix}$ and $\mathbf{T}=\begin{bmatrix}3 & 0\\ 0 & -1\end{bmatrix}$. Compute $\mathbf{ST}$ and $\mathbf{TS}$. What do you notice, and why is it a trap when you *test* code?

Part E • Identity, Inverse, Transpose

COMPUTE ≈ 6 min

Today you only *check* inverses by multiplying; computing them is Wednesday's job.

E1. A book claims $\begin{bmatrix}2 & 1\\ 5 & 3\end{bmatrix}^{-1}=\begin{bmatrix}3 & -1\\ -5 & 2\end{bmatrix}$. Check the claim, entry by entry.

E2. True or false? Multiply to decide, and say in words what each true inverse *does* to a shape.
(i) $\begin{bmatrix}4 & 0\\ 0 & 0.5\end{bmatrix}^{-1}=\begin{bmatrix}0.25 & 0\\ 0 & 2\end{bmatrix}$ (ii) $\begin{bmatrix}1 & 3\\ 0 & 1\end{bmatrix}^{-1}=\begin{bmatrix}-1 & -3\\ 0 & -1\end{bmatrix}$ (iii) $\begin{bmatrix}0 & 2\\ -1 & 0\end{bmatrix}^{-1}=\begin{bmatrix}0 & -1\\ 0.5 & 0\end{bmatrix}$

E3. Let $\mathbf{A}=\left[\begin{smallmatrix}2 & 1\\ 5 & 3\end{smallmatrix}\right]$ (E1) and $\mathbf{S}=\left[\begin{smallmatrix}4 & 0\\ 0 & 0.5\end{smallmatrix}\right]$ (E2(i)). Using only the inverses you already have, write $(\mathbf{AS})^{-1}$ as a single matrix, then check it against $\mathbf{AS}$.

E4. Someone uses $\mathbf{A}^{-1}\mathbf{S}^{-1}$ instead. Compute just the top-left entry of $(\mathbf{AS})(\mathbf{A}^{-1}\mathbf{S}^{-1})$ and say what went wrong.

E5. Write the transpose of $\begin{bmatrix}1 & 4 & 0\\ -2 & 5 & 3\end{bmatrix}$ and give its size. The entry 3 is a_{23} before; what is its name after?

E6. With $\mathbf{A}$ and $\mathbf{B}$ from Part D, compute $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ and compare it with the transpose of your answer to D3.

Part F • Two Ways to Read $\mathbf{M}\mathbf{x}$

TRACE ≈ 6 min

Let $\mathbf{M}=\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix}$ and $\mathbf{x}=(3,2)$.

F1. Row view. Compute $\mathbf{Mx}$ by dotting each row with $\mathbf{x}$.

F2. Column view. Write $\mathbf{Mx}$ as a mix of the columns, and draw it tip to tail on squared paper.

F3. Without multiplying: $\mathbf{M}(1,0)=$ ________ and $\mathbf{M}(0,1)=$ ________. Sketch the unit square's image.

F4. A transform sends $\mathbf{x}$ to $(1,2)$ and $\mathbf{y}$ to $(-2,1)$. Write its matrix $\mathbf{N}$ straight from that sentence, then find $\mathbf{N}(2,1)$ with the column view.

F5. Look at $\mathbf{N}$'s columns: their lengths, and their dot product. Is $\mathbf{N}$ orthogonal? Describe in words what it does to a shape, and check your description against $|\mathbf{N}|$.

Part G • Name That Matrix

CLASSIFY ≈ 5 min

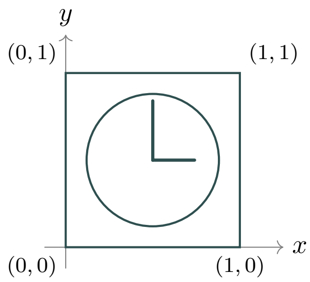
Tick every box that applies, give the determinant, and say in a few words what the matrix does to the clock.

matrix	diagonal	symmetric	orthogonal	det	what it does
(i) $\begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$	☐	☐	☐	______	______________________
(ii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	☐	☐	☐	______	______________________
(iii) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$	☐	☐	☐	______	______________________
(iv) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$	☐	☐	☐	______	______________________
(v) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$	☐	☐	☐	______	______________________
(vi) $\begin{bmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{bmatrix}$	☐	☐	☐	______	______________________

Part H • Scale and Shear the Clock

TRACE ≈ 7 min

The lecture's clock: a unit square with corners $(0,0)$, $(1,0)$, $(1,1)$, $(0,1)$; a round face of radius 0.38 centred at $(0.5,0.5)$; the **long hand** from the centre to $(0.5,0.84)$, pointing to 12; the **short hand** from the centre to $(0.74,0.5)$, pointing to 3. For every item, predict the picture in words *first*.



H1. Apply scale$(2,0.5)$. Where do the corners $(1,0)$, $(1,1)$, $(0,1)$ and the two hand tips go? What shape is the face, and what is the area of the new frame?

H2. Apply scale$(-1,1)$. Where do the centre and the two hand tips go? What time does the clock now show? Could *any* rotation produce this picture?

H3. Apply shear-$x(-0.5)$, that is $\begin{bmatrix} 1 & -0.5 \\ 0 & 1 \end{bmatrix}$. Follow $(0,1)$, $(1,1)$, the centre and the long-hand tip. Which way does the clock lean, and what is its area?

H4. Which shear tilts the clock's upright left edge to lean right so that its top corner lands at $(2,1)$? By what angle is the edge tilted, and how long is it now?

H5. Apply scale$(0,1)$. Describe the result, give its determinant, and explain why no matrix can undo it.

Part I • Rotation by Hand

COMPUTE ≈ 7 min

Use the angle ϕ with $\cos\phi = 0.8$ and $\sin\phi = 0.6$ ($\phi \approx 36.87°$). These numbers are exact, so no rounding is needed anywhere in this part.

I1. Write $\mathbf{R} = \text{rotate}(\phi)$. Check that both rows have length one and are at right angles.

I2. Compute $\mathbf{R}(5,0)$ and $\mathbf{R}(0,5)$. Why could you have written both answers down without multiplying?

I3. Rotate the clock's corner $(1,1)$. Check that its distance from the origin did not change.

I4. Undo it. Write rotate$(-\phi)$ using $\cos(-\phi) = \cos\phi$ and $\sin(-\phi) = -\sin\phi$, compare it with $\mathbf{R}^{\mathrm{T}}$, and apply it to $(0.2, 1.4)$.

I5. Without a calculator: which rotation is $\begin{bmatrix} 0.5 & 0.866 \\ -0.866 & 0.5 \end{bmatrix}$? Which way does it turn? Where does it send $(2,0)$?

I6. Compute $\mathbf{RR}$. Is the result a rotation? By what angle? Which trig facts have you just rediscovered?

I7. Put $\phi = 180°$ into rotate(ϕ). You get a matrix from Part G. H2 showed that flipping *one* axis is a mirror; why is flipping *both* a rotation?

Part J • Find the Bug

FIX ≈ 7 min

Each function is *almost* right. Matrices are lists of rows, `M[row][col]`; points are tuples. Name the bug and give the fix — test your fix on the numbers in the symptom.

J1. Symptom: `signed_area((4, 1), (1, 3))` returns `-11`.

```
def signed_area(a, b):
    return a[1] * b[0] - a[0] * b[1]
```

Bug & fix: ______________________________

J2. Symptom: `rot(0.5)` turns the clock *clockwise*.

```
def rot(phi):
    c, s = math.cos(phi), math.sin(phi)
    return [[c, s],
            [-s, c]]
```

Bug & fix: ______________________________

J3. Symptom: scale matrices work perfectly, but `transform([[1, -2], [2, 1]], (2, 1))` returns `(0, 1)`; F4 says $(0, 5)$.

```
def transform(M, p):
    x, y = p
    x = M[0][0] * x + M[0][1] * y
    y = M[1][0] * x + M[1][1] * y
    return (x, y)
```

Bug & fix: ______________________________

J4. Symptom: every test that multiplies two scale matrices passes, but `mat_mul(A, B)` with Part D's **A** and **B** returns `[[2, -2], [4, 2]]`.

```
def mat_mul(A, B):
    P = [[0, 0], [0, 0]]
    for i in range(2):
        for j in range(2):
            for k in range(2):
                P[i][j] += B[i][k] * A[k][j]
    return P
```

Bug & fix: ______________________________

J5. Symptom: `rot_deg(90)` turns the clock by about 116.6° — yet the clock is not squashed or stretched at all.

```
def rot_deg(deg):
    c, s = math.cos(deg), math.sin(deg)
    return [[c, -s], [s, c]]
```

Bug & fix: ____

Part K • One Sentence Each

EXPLAIN ≈ 4 min

K1. Why can no 2×2 matrix slide the clock one unit to the right?

K2. Why does scale$(3, 2)$ multiply the area of *every* shape by 6, not just the unit square?

K3. Why does a rotation never change area and never mirror a shape?

K4. Why does a shear keep the area, even though the shape changes?

K5. Why do a matrix's two columns tell you *everything* it does?

Part L • Bonus: A Rotation Made of Three Shears ★

DESIGN ≈ 10 min

For the brave. A shear looked like "rotating just one axis". It turns out that three shears, chosen well, make an *exact* rotation. Use the rotation $\mathbf{R}$ from Part I ($\cos\phi = 0.8$, $\sin\phi = 0.6$), and the half-angle fact $\tan\frac{\phi}{2} = \dfrac{\sin\phi}{1+\cos\phi}$.

L1. Compute $t = \tan\frac{\phi}{2}$. Then write $\mathbf{H} = \text{shear-}x(-t)$ and $\mathbf{V} = \text{shear-}y(\sin\phi)$ as matrices, and give both determinants.

L2. Compute $\mathbf{VH}$.

L3. Now compute $\mathbf{H}(\mathbf{VH})$. What do you get?

L4. Follow the corner $(1, 1)$ through the three shears one at a time ($\mathbf{H}$, then $\mathbf{V}$, then $\mathbf{H}$), and compare with I3. Why did you not have to worry about *which* shear acts first?

L5. Why would anyone rotate an *image* this way? Think about what $\mathbf{H}$ does to one row of pixels, and what $\mathbf{V}$ does to one column.

L6. Try the recipe for $\phi = 180°$. What breaks, and how would you rescue it using something from Part G?

Stuck on more than three items? Re-read the Lecture 7 slides — everything above Part L comes straight from them. Start with "Expand, and most terms vanish", "Left row meets right column" and "Columns show where $\mathbf{x}$ and $\mathbf{y}$ land"; most answers here are one of those three slides wearing different clothes.

Answer Key

Check your work *after* attempting everything. If an answer surprises you, that item is worth re-reading in the slides.

A1. the **signed area**. Positive for a **counterclockwise** turn, negative for a **clockwise** turn, zero when the two arrows lie on **one line** (collinear, including when one of them is the zero vector).

A2. $x_a y_b - y_a x_b$. Built from: **stretch** one side and the area stretches by the same factor; **slide** an edge along the other vector and the area stays the same; two areas on one base **add**. (Plus the unit-arrow facts $|\mathbf{xy}| = 1$, $|\mathbf{yx}| = -1$, $|\mathbf{vv}| = 0$.)

A3. $\mathbf{B}$ must have n **rows** (it is $n \times p$), and the product is $m \times p$. Entry p_{ij} walks along **row** i of $\mathbf{A}$ and down **column** j of $\mathbf{B}$, multiplying pairs and adding.

A4. **Not always** (order matters); = (grouping is free); = (both are $\mathbf{I}$ — one of the few orders that never matters).

A5. $\mathbf{B}^{-1}\mathbf{A}^{-1}$; $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$ (both reverse the order — shoes off before socks); $|\mathbf{A}|\,|\mathbf{B}|$; $1/|\mathbf{A}|$; $|\mathbf{A}|$.

A6. **Length one** and **at right angles** to each other. $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, so the inverse is the **transpose**, and $|\mathbf{R}| = \mathbf{+1}$ **or** $\mathbf{-1}$.

A7. $\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}$, $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ — cosines on the diagonal, *minus* sine top-right.

A8. the places where the unit arrows $\mathbf{x} = (1, 0)$ and $\mathbf{y} = (0, 1)$ **land**. No 2×2 matrix can **move the origin**: $\mathbf{M}(0, 0) = (0, 0)$ always.

B1. $|\mathbf{ab}| = (4)(3) - (1)(1) = 12 - 1 = \mathbf{11}$. Positive: $\mathbf{a}$ is the flatter arrow, and turning from it up to the steeper $\mathbf{b}$ is counterclockwise.

B2. The box is $5 \times 4 = 20$. The six pieces outside the parallelogram: bottom triangle $\frac{1}{2} \cdot 4 \cdot 1 = 2$, bottom-right square $1 \times 1 = 1$, right triangle $\frac{1}{2} \cdot 1 \cdot 3 = 1.5$, and the three matching pieces on top: 2, 1, 1.5. Total 9, and $20 - 9 = 11$. Same answer.

B3. $\mathbf{-11}$. Swapping the order reverses the turn, which flips the sign and keeps the size.

B4. $(2)(2) - (-1)(-4) = 4 - 4 = \mathbf{0}$. Because $\mathbf{d} = -2\mathbf{c}$: both arrows lie on one line through the origin, so the "parallelogram" is squashed flat and has no area.

B5. The classmate is wrong: the sign depends only on the *turn*, not on where the arrows point. $\mathbf{u}$ points down-left, $\mathbf{v}$ down-right; going from $\mathbf{u}$ to $\mathbf{v}$ through the smaller angle sweeps under the origin, counterclockwise. $(-2)(-3) - (-1)(2) = 6 + 2 = \mathbf{+8}$.

B6. $|\mathbf{ab}| = 3 \cdot 4 - k \cdot 6 = 12 - 6k$. Zero at $\mathbf{k = 2}$, when $\mathbf{a} = (3, 2)$ is exactly half of $\mathbf{b}$. Equal to 9 at $\mathbf{k = 0.5}$.

C1. **33** (stretch one side by 3, area by 3). $\mathbf{-22}$ (stretch by -1 and by 2: the factors multiply, $(-1)(2)(11)$).

C2. **11**. Adding a multiple of $\mathbf{a}$ to $\mathbf{b}$ slides the far edge *along* $\mathbf{a}$: same base, same height, same area. (The slide can be backwards, and as long as you like.)

C3. **44**. Each side stretches by 2, and each stretch multiplies the area by 2, so $2 \cdot 2 = 4$. Picture: doubling both sides of a parallelogram makes a 2×2 tiling of four copies.

C4. $|\mathbf{aa}| - |\mathbf{ab}| + |\mathbf{ba}| - |\mathbf{bb}| = 0 - 11 + (-11) - 0 = \mathbf{-22}$. The two squashed terms vanish, and $|\mathbf{ba}| = -|\mathbf{ab}|$. In general it is always $-2|\mathbf{ab}|$.

C5. $\mathbf{a} + \mathbf{b} = (5, 4)$ and $\mathbf{a} - \mathbf{b} = (3, -2)$. $(5)(-2) - (4)(3) = -10 - 12 = -22$. The rules and the formula agree — they have to, the formula *is* the rules.

D1. 2×1; **not defined** (2 columns meet 3 rows); 1×1, a single number (this is how a dot product is written); 2×2; 2×3. The two inner numbers must match, the two outer numbers survive.

D2. Row 1: $(1)(1) + (0)(2) + (2)(-1) = 1 + 0 - 2 = -1$. Row 2: $(-1)(1) + (3)(2) + (1)(-1) = -1 + 6 - 1 = 4$. Answer $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$, a 2×1 column.

D3. $p_{11} = (2)(1)+(1)(2) = 4$; $p_{12} = (2)(-1)+(1)(0) = -2$; $p_{21} = (0)(1)+(3)(2) = 6$; $p_{22} = (0)(-1)+(3)(0) = 0$. $\mathbf{AB} = \begin{bmatrix} 4 & -2 \\ 6 & 0 \end{bmatrix}$.

D4. $p_{11} = (1)(2)+(-1)(0) = 2$; $p_{12} = (1)(1)+(-1)(3) = -2$; $p_{21} = (2)(2)+(0)(0) = 4$; $p_{22} = (2)(1)+(0)(3) = 2$. $\mathbf{BA} = \begin{bmatrix} 2 & -2 \\ 4 & 2 \end{bmatrix}$ — **different**. Order matters.

D5. $|\mathbf{A}| = (2)(3) - (0)(1) = 6$; $|\mathbf{B}| = (1)(0) - (2)(-1) = 2$; $|\mathbf{AB}| = (4)(0) - (6)(-2) = 12$; $|\mathbf{BA}| = (2)(2) - (4)(-2) = 12$. Both are $|\mathbf{A}|\,|\mathbf{B}| = 6 \cdot 2$, and multiplying two *numbers* does not care about order, even though multiplying the matrices does.

D6. Both are $\begin{bmatrix} 6 & 0 \\ 0 & -5 \end{bmatrix}$: diagonal matrices just multiply their diagonals, and number multiplication commutes. The trap: a product written in the wrong order passes every test built from scale matrices. Test with a shear or a rotation — see J4.

E1. Row 1, col 1: $6 - 5 = 1$. Row 1, col 2: $-2 + 2 = 0$. Row 2, col 1: $15 - 15 = 0$. Row 2, col 2: $-5 + 6 = 1$. The product is $\mathbf{I}$, so the claim is **true**.

E2. (i) **True**: the product is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Undo a stretch by the reciprocal stretch. (ii) **False**: the product is $\begin{bmatrix} -1 & -6 \\ 0 & -1 \end{bmatrix}$. Negating every entry is not undoing. The real inverse is $\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$ — undo a shear by shearing back the other way; check: $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3+3 \\ 0 & 1 \end{bmatrix} = \mathbf{I}$. (iii) **True**: row 1 gives $(0)(0) + (2)(0.5) = 1$ and 0; row 2 gives 0 and $(-1)(-1) = 1$. Also $|\mathbf{M}| = 2$ and $|\mathbf{M}^{-1}| = 0.5 = 1/2$, as the rule says.

E3. $(\mathbf{AS})^{-1} = \mathbf{S}^{-1}\mathbf{A}^{-1} = \left[\begin{smallmatrix} 0.25 & 0 \\ 0 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 3 & -1 \\ -5 & 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0.75 & -0.25 \\ -10 & 4 \end{smallmatrix}\right]$. And $\mathbf{AS} = \left[\begin{smallmatrix} 8 & 0.5 \\ 20 & 1.5 \end{smallmatrix}\right]$. Check: $8(0.75)+0.5(-10) = 6-5 = 1$; $8(-0.25)+0.5(4) = 0$; $20(0.75)+1.5(-10) = 0$; $20(-0.25)+1.5(4) = -5+6 = 1$. $\mathbf{I}$.

E4. $\mathbf{A}^{-1}\mathbf{S}^{-1} = \left[\begin{smallmatrix} 3 & -1 \\ -5 & 2 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0.25 & 0 \\ 0 & 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0.75 & -2 \\ -1.25 & 4 \end{smallmatrix}\right]$. Top-left of the product: $8(0.75)+0.5(-1.25) = 6-0.625 = 5.375 \neq 1$. With the wrong order, $\mathbf{S}$ and $\mathbf{A}^{-1}$ end up in the middle, and nothing cancels.

E5. $\left[\begin{smallmatrix} 1 & -2 \\ 4 & 5 \\ 0 & 3 \end{smallmatrix}\right]$, a 3×2 matrix. Row 1 became column 1, row 2 became column 2. The 3 is now a'_{32} — the two indices swap.

E6. $\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}} = \left[\begin{smallmatrix} 1 & 2 \\ -1 & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 & 0 \\ 1 & 3 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 2+2 & 0+6 \\ -2+0 & 0+0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 4 & 6 \\ -2 & 0 \end{smallmatrix}\right]$, which is exactly $(\mathbf{AB})^{\mathrm{T}}$, the D3 answer $\left[\begin{smallmatrix} 4 & -2 \\ 6 & 0 \end{smallmatrix}\right]$ flipped across its diagonal. (The other order, $\mathbf{A}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} = \left[\begin{smallmatrix} 2 & 4 \\ -2 & 2 \end{smallmatrix}\right]$, does not match.)

F1. $\mathbf{r}_1 \cdot \mathbf{x} = (2)(3)+(-1)(2) = 6-2 = 4$; $\mathbf{r}_2 \cdot \mathbf{x} = (1)(3)+(1)(2) = 5$. $\mathbf{Mx} = (4,5)$.

F2. $3\,\mathbf{c}_1 + 2\,\mathbf{c}_2 = 3(2,1)+2(-1,1) = (6,3)+(-2,2) = (4,5)$. Three steps of $(2,1)$, then two steps of $(-1,1)$. Same point as F1: the same arithmetic, grouped down instead of across.

F3. $(2,1)$ and $(-1,1)$ — the two columns. The unit square becomes the parallelogram with corners $(0,0)$, $(2,1)$, $(1,2)$, $(-1,1)$, and its area is $|\mathbf{M}| = (2)(1)-(1)(-1) = 3$.

F4. $\mathbf{N} = \left[\begin{smallmatrix} 1 & -2 \\ 2 & 1 \end{smallmatrix}\right]$: the landing spots *are* the columns. $\mathbf{N}(2,1) = 2(1,2)+1(-2,1) = (2,4)+(-2,1) = (0,5)$.

F5. Both columns have length $\sqrt{1+4} = \sqrt{5}$, and $(1)(-2)+(2)(1) = 0$, so they are at right angles — but not length one, so $\mathbf{N}$ is **not orthogonal**. It turns a shape (the $\mathbf{x}$ arrow moves to angle $\approx 63.4^\circ$) and blows it up by $\sqrt{5}$ in every direction. Areas grow by $\sqrt{5}\cdot\sqrt{5} = 5$, and indeed $|\mathbf{N}| = (1)(1)-(2)(-2) = 5$.

G. (i) diagonal, symmetric; **not** orthogonal (column lengths 3 and 2). det $= -6$. Stretches x by 3, stretches y by 2 *and flips it*: a mirrored, enlarged clock.
(ii) symmetric, orthogonal (columns $(0,1)$, $(1,0)$: unit, at right angles). det $= -1$. Swaps x and y: a mirror image across the line $y = x$.
(iii) all three. det $= +1$. Every point goes to its opposite: this is rotate(180°), and also scale$(-1,-1)$.
(iv) none. det $= 1$. A vertical shear: points move *down* by twice their x, so the right edge slides down. Area unchanged.
(v) symmetric only. det $= 4-1 = 3$. A stretch that is not along the axes; areas triple.
(vi) orthogonal only. det $= 0.64+0.36 = 1$. A rotation; the bottom-left entry $\sin\phi = -0.6$ is negative, so it turns *clockwise*, by $\approx 36.9^\circ$.
Lesson from (i)–(iii): every diagonal matrix is symmetric, but an orthogonal matrix is not always a rotation — (ii) has det $= -1$ and is a mirror.

H1. Corners: $(2,0)$, $(2,0.5)$, $(0,0.5)$. Long-hand tip $(1,0.42)$, short-hand tip $(1.48,0.25)$, centre $(1,0.25)$. The face becomes an ellipse, 0.76 across its half-width and 0.19 up. The frame is a 2×0.5 rectangle with area 1 — matching $|\mathbf{M}| = 2\cdot 0.5 = 1$. Squashed, but no bigger.

H2. Centre $(-0.5,0.5)$, long tip $(-0.5,0.84)$, short tip $(-0.74,0.5)$. The long hand still points up but the short hand now points *left*: the clock reads **9 o'clock**. No rotation can do this: on the real clock you turn *clockwise* from 12 to the short hand; on the new one you turn counterclockwise. That turn-reversal is a mirror, and it shows up as $|\mathbf{M}| = -1$. Every rotation has determinant $+1$.

H3. $(0,1)\to(-0.5,1)$; $(1,1)\to(0.5,1)$; centre $(0.5,0.5)\to(0.25,0.5)$; long tip $(0.5,0.84)\to(0.08,0.84)$. The bottom edge stays put, and every point moves *left* by half its height: the clock leans **left**. Area 1 — same base, same height, and $|\mathbf{M}| = 1$.

H4. $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$, shear-$x(2)$: $(0,1)\to(2,1)$. This is $\left[\begin{smallmatrix} 1 & \tan\phi \\ 0 & 1 \end{smallmatrix}\right]$ with $\tan\phi = 2$, so the edge tilts clockwise by $\phi \approx 63.4^\circ$. Its length is now $\sqrt{4+1} = \sqrt{5} \approx 2.24$ — a shear only *looks* like turning one axis; a real rotation would keep the length 1.

H5. Every point (x,y) goes to $(0,y)$: the whole clock is squashed onto the segment from $(0,0)$ to $(0,1)$. $|\mathbf{M}| = 0$ — no area left. It cannot be undone: $(0.2,0.5)$ and $(0.9,0.5)$ both land on $(0,0.5)$, so no matrix could know which one to send back. (The rule $|\mathbf{M}^{-1}| = 1/|\mathbf{M}|$ already warned you: 1/0 does not exist.)

I1. $\mathbf{R} = \left[\begin{smallmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{smallmatrix}\right]$. Row lengths squared: $0.64+0.36 = 1$ for both. Dot product of the rows: $(0.8)(0.6)+(-0.6)(0.8) = 0.48-0.48 = 0$. Orthogonal, as every rotation is.

I2. $\mathbf{R}(5,0) = (4,3)$ and $\mathbf{R}(0,5) = (-3,4)$. $(5,0)$ is $5\mathbf{x}$, so it lands on 5 times the first column; $(0,5)$ lands on 5 times the second column. Both still have length 5 (3-4-5 triangles).

I3. Row 1: $(0.8)(1)+(-0.6)(1) = 0.2$. Row 2: $(0.6)(1)+(0.8)(1) = 1.4$. So $(0.2,1.4)$. Length squared: $0.04+1.96 = 2$, the same as 1^2+1^2.

I4. rotate$(-\phi) = \left[\begin{smallmatrix} 0.8 & 0.6 \\ -0.6 & 0.8 \end{smallmatrix}\right]$, which *is* $\mathbf{R}^{\mathrm{T}}$ — the minus sign moved to the bottom-left. Applied: $(0.8)(0.2)+(0.6)(1.4) = 0.16+0.84 = 1$ and $(-0.6)(0.2)+(0.8)(1.4) = -0.12+1.12 = 1$. Back to $(1,1)$. Undoing a turn is turning back, and it costs no inverse, just a transpose.

I5. $\cos\phi = 0.5$ and $\sin\phi = -0.866$ (bottom-left entry), so $\phi = -60^\circ$: **clockwise** by 60°. $(2,0)$ lands on twice the first column, $(1,-1.732)$ — below the x-axis, as a clockwise turn should be.

I6. $\left[\begin{smallmatrix} 0.64-0.36 & -0.48-0.48 \\ 0.48+0.48 & -0.36+0.64 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0.28 & -0.96 \\ 0.96 & 0.28 \end{smallmatrix}\right]$. It has the rotation pattern, with cos $= 0.28$ and sin $= 0.96$ $(0.0784+0.9216 = 1)$: it is rotate(2ϕ), $\approx 73.74^\circ$. Turning by ϕ twice is turning by 2ϕ, which is the double-angle rules $\cos 2\phi = \cos^2\phi - \sin^2\phi$ and $\sin 2\phi = 2\sin\phi\cos\phi$, read off a matrix product.

I7. rotate$(180^\circ) = \left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right] =$ scale$(-1,-1)$, G(iii). Each single flip reverses the turn direction (det $= -1$); doing two reverses it back (det $= (-1)(-1) = +1$). Two mirrors make a turn. Check on the clock: long hand points down (to 6), short hand points left (to 9) — a clock turned upside down, not a mirrored one.

J1. The two products are subtracted in the wrong order: this is $y_a x_b - x_a y_b = |\mathbf{ba}|$. Every counterclockwise pair is reported as clockwise. Fix: `return a[0] * b[1] - a[1] * b[0]` — x of the first times y of the second, first. B1 gives +11.

J2. The minus sign is bottom-left instead of top-right. This is $\text{rotate}(\phi)^{\mathrm{T}} = \text{rotate}(-\phi)$ — the *inverse*. (It is also what you get by copying a matrix from a source that writes vectors as rows.) Fix: `[[c, -s], [s, c]]`. Quick test: the bottom-left entry must be $\sin\phi$, positive for a small positive angle.

J3. Line two overwrites `x`, so line three uses the *new* x (0) instead of the original (2): $y = 2 \cdot 0 + 1 \cdot 1 = 1$. Both rows must read the *old* point. Fix: `nx = ...`, `ny = ...`, `return (nx, ny)`. Scales hide it because their `M[1][0]` is 0, so the spoiled x gets multiplied by zero.

J4. It walks row i of $\mathbf{B}$ down column j of $\mathbf{A}$, which computes $\mathbf{BA}$ — exactly the D4 answer. Fix: `P[i][j] += A[i][k] * B[k][j]`. The scale tests stayed silent because diagonal matrices commute (D6): for them the wrong order gives the right answer.

J5. `math.cos` and `math.sin` want *radians*; 90 radians is 116.6° past a whole number of turns. Fix: `r = math.radians(deg)` and use `r`. The clock keeps its shape because cosine and sine of *any* single number still give a perfect rotation matrix. Rule of thumb: a distorted picture means a matrix bug; a perfect picture at the wrong angle means a units bug.

K1. Any matrix times $(0, 0)$ is $(0, 0)$, so the corner at the origin can never move — and sliding moves every point. (Wednesday fixes this with an extra coordinate.)

K2. Any shape can be cut into tiny squares, each tiny square is stretched by the same matrix into a rectangle 6 times its area, and $|\mathbf{M}| = 3 \cdot 2 = 6$ is exactly that factor.

K3. Its determinant is $\cos^2\phi + \sin^2\phi = 1$: the size of the area factor is 1, and the sign is +.

K4. It slides each line of points sideways by an amount that depends only on its height, so the base and the height are unchanged — the "sliding the top edge" rule, and $|\mathbf{M}| = 1$.

K5. Every input is $(x_1, x_2) = x_1\mathbf{x} + x_2\mathbf{y}$, so its image is $x_1\mathbf{c}_1 + x_2\mathbf{c}_2$: once you know where $\mathbf{x}$ and $\mathbf{y}$ land, every other point follows.

L1. $t = 0.6/1.8 = \frac{1}{3}$. $\mathbf{H} = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 1 \end{bmatrix}$, $\mathbf{V} = \begin{bmatrix} 1 & 0 \\ 0.6 & 1 \end{bmatrix}$. Both have determinant 1: shears never change area.

L2. $p_{11} = 1$; $p_{12} = (1)(-\frac{1}{3}) + (0)(1) = -\frac{1}{3}$; $p_{21} = (0.6)(1) + (1)(0) = 0.6$; $p_{22} = (0.6)(-\frac{1}{3}) + (1)(1) = -0.2 + 1 = 0.8$. $\mathbf{VH} = \begin{bmatrix} 1 & -\frac{1}{3} \\ 0.6 & 0.8 \end{bmatrix}$.

L3. $p_{11} = 1 + (-\frac{1}{3})(0.6) = 1 - 0.2 = 0.8$; $p_{12} = -\frac{1}{3} + (-\frac{1}{3})(0.8) = -\frac{1.8}{3} = -0.6$; $p_{21} = 0.6$; $p_{22} = 0.8$. $\mathbf{HVH} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} = \mathbf{R}$, **exactly**. Its determinant is $1 \cdot 1 \cdot 1 = 1$, as a rotation's must be.

L4. $\mathbf{H}$: $(1 - \frac{1}{3},\ 1) = (\frac{2}{3}, 1)$. $\mathbf{V}$: $(\frac{2}{3},\ 0.6 \cdot \frac{2}{3} + 1) = (\frac{2}{3}, 1.4)$. $\mathbf{H}$: $(\frac{2}{3} - \frac{1.4}{3},\ 1.4) = (0.2, 1.4)$. Exactly I3's answer. The product $\mathbf{HVH}$ reads the same forwards and backwards, so the order question never comes up. For a general chain of moves it does — that is Wednesday.

L5. $\mathbf{H}$ keeps every y, so each row of pixels just slides sideways as a unit; $\mathbf{V}$ keeps every x, so each column slides up or down. Three passes of one-dimensional sliding replace a full 2D rotation, and because every pass has determinant 1, no pass stretches the picture or opens gaps between pixels. This trick (Paeth, 1986) is how many image programs rotated pictures quickly.

L6. $\tan 90°$ does not exist ($1 + \cos\phi = 0$), so $\mathbf{H}$ would need an infinite shear; angles near 180° already need huge ones. Rescue: first apply $\text{rotate}(180°) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, G(iii), which just flips the image both ways with no arithmetic at all, then shear-rotate by the small remaining angle $\phi - 180°$.